Math 113 Calculus – Midterm Exam 1 – Solutions

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Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Also note that “Yolum doğru, bir tek işlem hatası yapmışım.” translates as “I apologize for writing this garbage!” Your answer is credited up to the first instance where you make a mistake. The rest is considered a garbage and your apology is accepted.

You should strive for excellence as if it is essential, because in the final analysis it is.

Q-1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Assume that $f'$ is not continuous at $x = 0$.

Prove or disprove the following statement:

It is possible that $\lim_{x \to 0^+} f'(x) = L$ for some $L > f'(0)$.

Solution:

This is not possible; it violates the intermediate value property of the derivative.

Assume to the contrary that $\lim_{x \to 0^+} f'(x) = L$ for some $L > f'(0)$.

Since $L > f'(0)$, we can find an $\epsilon > 0$ and $K \in \mathbb{R}$ such that $f'(0) < K < L - \epsilon$.

Since $\lim_{x \to 0^+} f'(x) = L$, for the above $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (0, \delta)$ we must have $K < L - \epsilon < f'(x)$.

In particular, pick any $x_0 \in (0, \delta)$. While we clearly have $f'(0) < K < L - \epsilon < f'(x_0)$, there is no $x \in (0, x_0)$ satisfying $f'(x) = K$. This violates the intermediate value property of the derivative.
Q-2) Write your answers to the space provided. No partial credits.

• \( f(x) = x^x, \ f'(x) = x^x(\ln x + 1). \)

• \( f(x) = x^\pi - \pi^x, \ f'(x) = \pi x^{\pi - 1} - \pi^x(\ln \pi). \)

• \( f(x) = \arctan[x + \ln(x^3 - 1)], \ f'(x) = \frac{1}{1 + [x + \ln(x^3 - 1)]^2} \cdot (1 + \frac{1}{x^3 - 1} \cdot (3x^2)). \)

• \( f(0) = 5, \ f'(0) = 10, \ f(7) = -3, \ f'(7) = -6, \ g(0) = 7, \ g'(0) = 8, \ g(7) = 11, \ g'(7) = 22 \)

\[
\lim_{x \to 0} \frac{f(g(x)) - f(g(0))}{x} = (f \circ g)'(0) = f'(g(0))g'(0) = f'(7) \cdot 8 = -6 \cdot 8 = -48.
\]
Q-3) Let \( y = f(x) \) be a curve in \( \mathbb{R}^2 \), where \( f \) is a differentiable function. Let \( p_0 = (x_0, y_0) \in \mathbb{R}^2 \) be a point not on the curve. Let \( y = L(x) \) be a line through \( p_0 \) and tangent to the curve at \( p_1 = (x_1, y_1) \) on the curve. Describe how you solve for \( x_1 \).

Let \( f(x) = x^2 \) and \( p_0 = (e - 1/2, 0) \) in the above set up. Find \( x_1 > 1 \) as you described for the above question. (In this case you may have to solve a certain equation by inspection if analytic solution looks too complicated, because if you have to solve an equation, then anything goes!)

Solution:

A line through the points \( p_0 = (x_0, y_0) \) and \( p_1 = (x_1, y_1) = (x_1, f(x_1)) \) has the slope \( m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - y_0}{x_1 - x_0} \).

On the other hand, if this line is tangent to the curve at the point \( p_1 \), then its slope is \( m = f'(x_1) \).

Equating the two values of \( m \) we get \( f'(x_1) = \frac{f(x_1) - y_0}{x_1 - x_0} \). We then solve this for \( x_1 \).

When \( (x_0, y_0) = (e - 1/2, 0) \) and \( f(x) = x^2 \), the equation to solve becomes:

\[
(ln x + 1)(x - e + \frac{1}{2}) = 1
\]

one of whose solutions is readily seen to be \( x = e \). There is another solution at \( x = 0.2228735\ldots \).

Here is the graph of \( y = (\ln x + 1)(x - e + \frac{1}{2}) - 1 \):

![Graph of the function](image-url)
Q-4) Show that \( f(x) = \frac{\ln x}{x} \) is increasing on \((0, e)\) and decreasing on \((e, \infty)\).

Explain which of the following holds:

(i) \( e^\pi > \pi^e \)
(ii) \( e^\pi < \pi^e \)
(iii) \( e^\pi = \pi^e \)

Solution:

\[ f'(x) = \frac{1 - \ln x}{x^2}, \]
so its sign depends only on the sign of \( 1 - \ln x \) which is positive on \((0, e)\) and negative on \((e, \infty)\), thus the answer.

Below is a graph of \( f(x) = \frac{\ln x}{x} \).

For \( x_1, x_2 \in [e, \infty) \) with \( x_1 < x_2 \) we must have \( \frac{\ln x_1}{x_1} > \frac{\ln x_2}{x_2} \) since \( f \) is decreasing there. Taking \( x_1 = e \) and \( x_2 = \pi \), we get

\[ \frac{\ln e}{e} > \frac{\ln \pi}{\pi} \quad \text{or equivalently} \quad \pi \ln e > e \ln \pi \quad \text{or equivalently} \quad \ln e^\pi > \ln \pi^e. \]

Since \( \ln \) is one-to-one and increasing, we must then have

\[ e^\pi > \pi^e. \]

In fact \( e^\pi \approx 23.14 \) and \( \pi^2 = 22.45 \).
Q-5) Prove or disprove the following statement:

The plane curve \( y = xe^{-\sin x} \) has only finitely many horizontal tangents.

Solution:

This is false. We can see it as follows:

If \( y = xe^{-\sin x} \), then \( y' = e^{-\sin x}(1 - x \cos x) = 0 \) when \( \cos x = \frac{1}{x} \).

Since \( \cos x \) oscillates between -1 and +1 infinitely many times on \((0, \infty)\) and since \( 0 < \frac{1}{x} < 1 \) for all \( x \in (1, \infty) \), these two curves intersect infinitely many times on \((0, \infty)\) giving infinitely many horizontal tangent lines for our function \( y \).

On the left below is a graph of \( y = xe^{-\sin x} \), and on the right is a graph showing the intersections of \( \cos x \) and \( \frac{1}{x} \):

In fact here is the graph of the derivative \( y' \), where you can see the infinitely many vanishing points.