Date: 27 October 2010, Wednesday NAME: Carl F. Gauß

Time: 18:00-20:00

Ali Sinan Sertöz STUDENT NO: 1

**Math 113 Calculus – Midterm Exam 1 – Solutions**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| 20 | 20 | 20 | 20 | 20 | 100 |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
**Also note that** "Yolum doğru, bir tek işlem hatası yapmışım." translates as "I apologize for writing this garbage!" Your answer is credited up to the first instance where you make a mistake. The rest is considered a garbage and your apology is accepted.

*You should strive for excellence as if it is essential, because in the final analysis it is.*

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**Q-1)** Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function. Assume that $f'$ is not continuous at $x=0$.
Prove or disprove the following statement:

$$\text{It is possible that } \lim_{x\to 0^+} f'(x) = L \text{ for some } L > f'(0).$$

**Solution:**

This is not possible; it violates the intermediate value property of the derivative.

Assume to the contrary that $\lim\limits_{x\to 0^+} f'(x) = L$ for some $L > f'(0)$.

Since $L > f'(0)$, we can find an $\epsilon > 0$ and $K \in \mathbb{R}$ such that $f'(0) < K < L - \epsilon$.

Since $\lim_{x\to 0^+} f'(x) = L$, for the above $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (0,\delta)$ we must have $K < L - \epsilon < f'(x)$.

In particular, pick any $x_0 \in (0,\delta)$. While we clearly have $f'(0) < K < L - \epsilon < f'(x_0)$, there is no $x \in (0, x_0)$ satisfying $f'(x) = K$. This violates the intermediate value property of the derivative.

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**Q-2)** Write your answers to the space provided. No partial credits.

- $f(x) = x^x$, $f'(x) = x^x(\ln x + 1)$.
- $f(x) = x^\pi - \pi^x$, $f'(x) = \pi x^{\pi-1} - \pi^x(\ln \pi)$.
- $f(x) = \arctan[x + \ln(x^3 - 1)]$, $f'(x) = \dfrac{1}{1 + [x + \ln(x^3 - 1)]^2} \cdot (1 + \dfrac{1}{x^3 - 1} \cdot (3x^2))$.
- $f(0) = 5,\ f'(0) = 10,\ f(7) = -3,\ f'(7) = -6,\ g(0) = 7,\ g'(0) = 8,\ g(7) = 11,\ g'(7) = 22$

$$\lim_{x\to 0} \frac{f(g(x)) - f(g(0))}{x} = (f \circ g)'(0) = f'(g(0))g'(0) = f'(7) \cdot 8 = -6 \cdot 8 = -48.$$

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**Q-3)** Let $y = f(x)$ be a curve in $\mathbb{R}^2$, where $f$ is a differentiable function. Let $p_0 = (x_0, y_0) \in \mathbb{R}^2$ be a point not on the curve. Let $y = L(x)$ be a line through $p_0$ and tangent to the curve at $p_1 = (x_1, y_1)$ on the curve. Describe how you solve for $x_1$.

Let $f(x) = x^x$ and $p_0 = (e - 1/2,\, 0)$ in the above set up. Find $x_1 > 1$ as you described for the above question. (In this case you may have to solve a certain equation by inspection if analytic solution looks too complicated, *because if you have to solve an equation, then anything goes!*)

**Solution:**

A line through the points $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1) = (x_1, f(x_1))$ has the slope $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{f(x_1) - y_0}{x_1 - x_0}$.

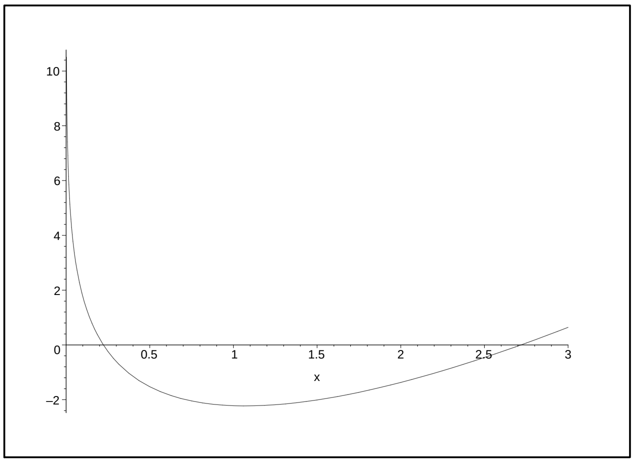
On the other hand, if this line is tangent to the curve at the point $p_1$, then its slope is $m = f'(x_1)$.

Equating the two values of $m$ we get $f'(x_1) = \dfrac{f(x_1) - y_0}{x_1 - x_0}$. We then solve this for $x_1$.

When $(x_0, y_0) = (e - 1/2,\, 0)$ and $f(x) = x^x$, the equation to solve becomes:

$$(\ln x + 1)(x - e + \frac{1}{2}) = 1$$

one of whose solutions is readily seen to be $x = e$. There is another solution at $x = 0.2228735....$ Here is the graph of $y = (\ln x + 1)(x - e + \frac{1}{2}) - 1$:



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**Q-4)** Show that $f(x) = \dfrac{\ln x}{x}$ is increasing on $(0, e)$ and decreasing on $(e, \infty)$.

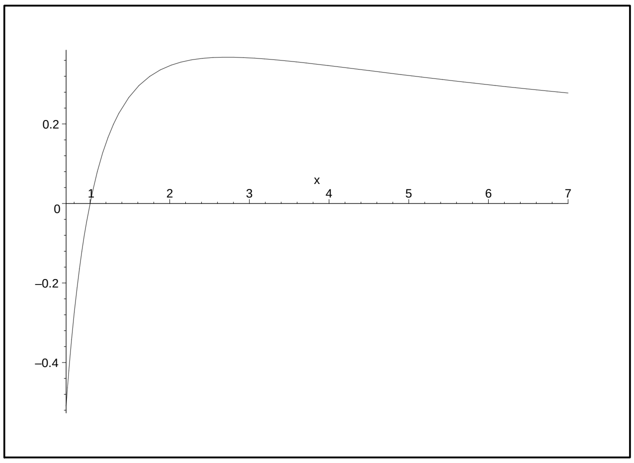
Explain which of the following holds:

(i) $e^\pi > \pi^e$ (ii) $e^\pi < \pi^e$ (iii) $e^\pi = \pi^e$

**Solution:**

$f'(x) = \dfrac{1 - \ln x}{x^2}$, so its sign depends only on the sigh of $1 - \ln x$ which is positive on $(0, e)$ and negative on $(e, \infty)$, thus the answer.

Below is a graph of $f(x) = \dfrac{\ln x}{x}$.




For $x_1, x_2 \in [e, \infty)$ with $x_1 < x_2$ we must have $\dfrac{\ln x_1}{x_1} > \dfrac{\ln x_2}{x_2}$ since $f$ is decreasing there. Taking $x_1 = e$ and $x_2 = \pi$, we get

$$\frac{\ln e}{e} > \frac{\ln \pi}{\pi} \quad \text{or equivalently} \quad \pi \ln e > e \ln \pi \quad \text{or equivalently} \quad \ln e^\pi > \ln \pi^e.$$

Since $\ln$ is one-to-one and increasing, we must then have

$$e^\pi > \pi^e.$$

In fact $e^\pi \approx 23.14$ and $\pi^2 = 22.45$.

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**Q-5)** Prove or disprove the following statement:

The plane curve $y = xe^{-\sin x}$ has only finitely many horizontal tangents.

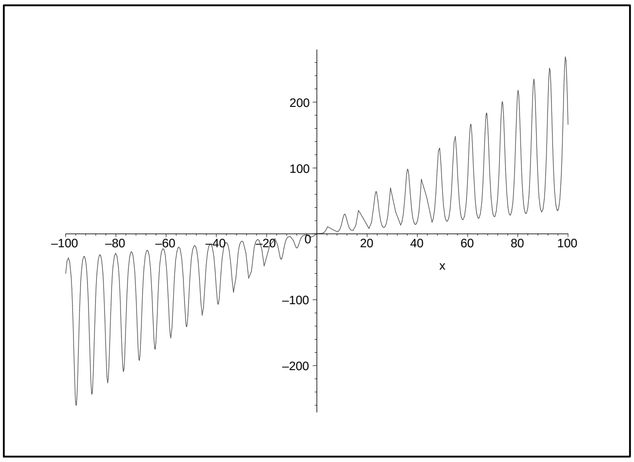
**Solution:**

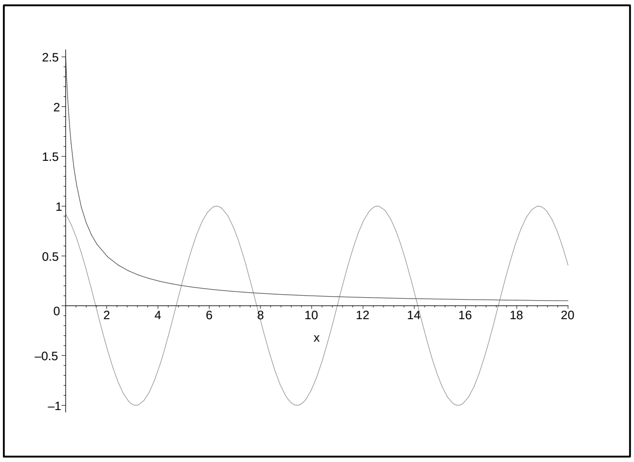
This is false. We can see it as follows:

If $y = xe^{-\sin x}$, then $y' = e^{-\sin x}(1 - x\cos x) = 0$ when $\cos x = \frac{1}{x}$.

Since $\cos x$ oscillates between -1 and +1 infinitely many times on $(0, \infty)$ and since $0 < \frac{1}{x} < 1$ for all $x \in (1, \infty)$, these two curves intersect infinitely many times on $(0, \infty)$ giving infinitely many horizontal tangent lines for our function $y$.

On the left below is a graph of $y = xe^{-\sin x}$, and on the right is a graph showing the intersections of $\cos x$ and $\frac{1}{x}$:




In fact here is the graph of the derivative $y'$, where you can see the infinitely many vanishing points.