Date: 16 December 2010, Thursday NAME:.....................................................
Time: 18:00-20:00 STUDENT NO:.....................................................

Math 113 Calculus – Midterm Exam 2 – Solutions

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Please do not write anything inside the above boxes!

Strive for excellence as if it is essential, because in the final analysis it is.

Q-1) Find the limit \( \lim_{n \to \infty} \sum_{k=0}^{n} \frac{n}{n^2 + k^2} \).

Solution:

First observe that

\[
\sum_{k=0}^{n} \frac{n}{n^2 + k^2} = \sum_{k=0}^{n} \frac{1}{n} \frac{1}{1 + (k/n)^2} = \frac{1}{2n} \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2}.
\]

Consider the upper Riemann sum \( UR(f, P_n) \) for the function \( f(x) = \frac{1}{1 + x^2} \), on the interval \([0, 1]\) for the partition \( P_n = \{ \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} \} \):

\[
UR(f, P_n) = \sum_{k=0}^{n-1} \frac{1}{n} \frac{1}{1 + (k/n)^2}.
\]

When \( n \) goes to infinity, the norm of the partition goes to zero and, since \( f \) is continuous on the interval, the limit is the integral of \( f \) on \([0, 1]\):

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{n}{n^2 + k^2} = \lim_{n \to \infty} \left( \frac{1}{2n} + \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2} \right) = \int_{0}^{1} \frac{1}{1 + x^2} \, dx = \left( \arctan x \right|_{0}^{1} = \frac{\pi}{4}.
\]
Q-2) Write your answers to the space provided. No partial credits.

- \( f(x) = x^{\cos x}, \quad f'(x) = x^{\cos x} \left( -\sin x \ln x + \frac{\cos x}{x} \right) \)

- \( f(x) = \tan(x^x - \pi^x), \quad f'(x) = \sec^2(x^x - \pi^x) (\pi x^{x-1} + \pi^x \ln \pi) \)

- \( f(x) = (\ln(x^3 + 7x - 1))^4, \quad f'(x) = \frac{4(\ln(x^3 + 7x - 1))^3 (3x^2 + 7)}{x^3 + 7x - 1} \)

- \( f(x) = x^4(\tan x^2)^3, \quad f'(x) = 4x^3(\tan x^2)^3 + x^4(3(\tan x^2)^2 \sec^2 x^2 2x) \)

- \( f(0) = 5, \quad f'(0) = 10, \quad f(5) = -3, \quad f'(5) = -6, \quad g(0) = 7, \quad g'(0) = 8, \quad g(5) = 11, \quad g'(5) = 22 \)

\[
\lim_{x \to 0} \frac{g(f(x)) - g(f(0))}{x} = (g \circ f)'(0) = g'(f(0)) f'(0) = g'(5)f'(0) = 22 \cdot 10 = 220.
\]

\[
\lim_{x \to 0} \frac{f(g(x)) - f(g(0))}{x} = (f \circ g)'(0) = f'(g(0)) g'(0) = f'(7)g'(0) = f'(7) \cdot 8.
\]
Q-3) We have a factory located at A. We want to transfer our goods to location D. There is a railroad along the line BD, where B is the foot of the perpendicular from our factory to the railroad. The distance from our factory to the railroad is 5km. The distance from B to D is 12km. The cost of transport by truck along open field is $\alpha$ TL/km, and the cost of transport by railroad is $\beta$ TL/km. TCDD agrees to build a station wherever we want. We want to find the location of the station C so that the cost of transport is minimized by carrying our goods from A to C by truck and loading them to train to be carried to D.

![Diagram of A, B, C, and D]

i) Solve the problem for $\alpha = 5$, $\beta = 3$. (15 points)

ii) For which values of $\alpha > 0$ and $\beta > 0$, the solution will be $C = B$? (5 points)

Solution: Let $BC = x$. The function to minimize is

$$f(x) = \alpha\sqrt{25 + x^2} + (12 - x)\beta, \quad x \in [0, 12].$$

Its derivative is

$$f'(x) = \frac{\alpha x}{\sqrt{25 + x^2}} - \beta.$$

In particular $f'(0) = -\beta < 0$, so the minimum never occurs at $x = 0$. This answers the second part.

When $\alpha = 5$, $\beta = 3$, the solution of $f'(x) = 0$ is at $15/4 \in [0, 12]$. To decide if this gives the minimum value or not we can do one of two things.

We either notice that

$$f''(x) = \frac{25\alpha}{(25 + x^2)^{3/2}} > 0, \text{ for } x \in [0, 12]$$

and conclude that $x = 15/4$ gives the minimum value,

or

we evaluate $f$ at $x = 15/4$ and also at the end points

$$f(0) = 61, \quad f(15/4) = 56, \quad f(12) = 65,$$

and conclude that $x = 15/4 = 3.75$ gives the minimum.
Q-4) Evaluate the integral
\[ \int_0^{\pi/3} \sec^7 x \, dx, \]
assuming that \( \int_0^{\pi/3} \sec^6 x \, dx \approx 25/3 \) and \( \int_0^{\pi/3} \sec^5 x \, dx \approx 21/4. \)

Solution: We first try integration by parts by choosing \( u = \sec^5 x \) and \( dv = \sec^2 x \, dx. \) This gives
\[
\int \sec^7 x \, dx = \sec^5 x \tan x - 5 \int \sec^5 x \tan^2 x \, dx
= \sec^5 x \tan x - 5 \int \sec^5 x (\sec^2 x - 1) \, dx
= \sec^5 x \tan x - 5 \int \sec^7 x \, dx + 5 \int \sec^5 x \, dx,
\]
giving us
\[
\int \sec^7 x \, dx = \frac{1}{6} \sec^5 x \tan x + \frac{5}{6} \int \sec^5 x \, dx.
\]
Finally, evaluating this from 0 to \( \pi/3 \) we get
\[
\int \sec^7 x \, dx = 13.618, \text{ or using the above approximation } \approx \frac{16}{\sqrt{3}} + \frac{35}{8} = 13.612.
\]
Q-5) Evaluate the integral
\[
\int \frac{x}{x^2 + 2x + 5} \, dx.
\]

Solution:
\[
\frac{x}{x^2 + 2x + 5} = \frac{1}{2} \frac{(2x + 2) - 2}{x^2 + 2x + 5} = \frac{1}{2} \frac{(2x + 2)}{x^2 + 2x + 5} - \frac{1}{x^2 + 2x + 5}
\]
\[
= \frac{1}{2} \frac{(2x + 2)}{x^2 + 2x + 5} - \frac{1}{4} \frac{1}{(\frac{x+1}{2})^2 + 1}
\]
\[
\frac{dx}{x^2 + 2x + 5} = \frac{1}{2} \frac{d(x^2 + 2x + 5)}{x^2 + 2x + 5} - \frac{1}{2} \frac{d(\frac{x+1}{2})}{(\frac{x+1}{2})^2 + 1}
\]
\[
\int \frac{x}{x^2 + 2x + 5} \, dx = \frac{1}{2} \ln(x^2 + 2x + 5) - \frac{1}{2} \arctan\left(\frac{x + 1}{2}\right) + C.
\]