Date: 16 December 2010, Thursday
Time: 18:00-20:00
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 113 Calculus – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Strive for excellence as if it is essential, because in the final analysis it is.

Q-1) Find the limit $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n}{n^2+k^2}$.

Solution:

First observe that

$$\sum_{k=0}^{n} \frac{n}{n^2+k^2} = \sum_{k=0}^{n} \frac{1}{n}\frac{1}{1+(k/n)^2} = \frac{1}{2n} + \sum_{k=0}^{n-1} \frac{n}{n^2+k^2}.$$

Consider the upper Riemann sum $UR(f, P_n)$ for the function $f(x) = \dfrac{1}{1+x^2}$, on the interval $[0, 1]$ for the partition $P_n = \{\frac{1}{n}, \dots, \frac{k}{n}, \dots, \frac{n}{n}\}$:

$$UR(f, P_n) = \sum_{k=0}^{n-1} \frac{1}{n}\frac{1}{1+(k/n)^2}.$$

When n goes to infinity, the norm of the partition goes to zero and, since f is continuous on the interval, the limit is the integral of f on $[0, 1]$;

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \left(\frac{1}{2n} + \sum_{k=0}^{n-1} \frac{n}{n^2+k^2} \right) = \int_0^1 \frac{1}{1+x^2}\, dx = \left(\arctan x \Big|_0^1 \right) = \frac{\pi}{4}.$$

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Q-2) Write your answers to the space provided. No partial credits.

- $f(x) = x^{\cos x}$, $f'(x) = x^{\cos x}\left(-\sin x\ \ln x + \frac{\cos x}{x}\right)$
- $f(x) = \tan(x^{\pi} - \pi^{x})$, $f'(x) = \sec^2(x^{\pi} - \pi^{x})\left(\pi x^{\pi-1} + \pi^{x}\ln\pi\right)$
- $f(x) = \left(\ln(x^3+7x-1)\right)^4$, $f'(x) = \dfrac{4\left(\ln(x^3+7x-1)\right)^3(3x^2+7)}{x^3+7x-1}$.
- $f(x) = x^4(\tan x^2)^3$, $f'(x) = 4x^3(\tan x^2)^3 + x^4(3(\tan x^2)^2\ \sec^2 x^2\, 2x)$.
- $f(0)=5,\ f'(0)=10,\ f(5)=-3,\ f'(5)=-6,\ g(0)=7,\ g'(0)=8,\ g(5)=11,\ g'(5)=22$

$$\lim_{x\to 0}\frac{g(f(x))-g(f(0))}{x} = (g\circ f)'(0) = g'(f(0))\,f'(0) = g'(5)f'(0) = 22\cdot 10 = 220.$$

$$\lim_{x\to 0}\frac{f(g(x))-f(g(0))}{x} = (f\circ g)'(0) = f'(g(0))\,g'(0) = f'(7)g'(0) = f'(7)\cdot 8.$$

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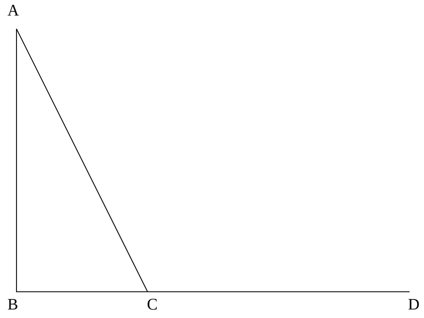
Q-3) We have a factory located at **A**. We want to transfer our goods to location **D**. There is a railroad along the line **BD**, where **B** is the foot of the perpendicular from our factory to the railroad. The distance from our factory to the railroad is 5km. The distance from **B** to **D** is 12km. The cost of transport by truck along open field is α TL/km, and the cost of transport by railroad is β TL/km. TCDD agrees to build a station wherever we want. We want to find the location of the station **C** so that the cost of transport is minimized by carrying our goods from **A** to **C** by truck and loading them to train to be carried to **D**.



i) Solve the problem for $\alpha = 5$, $\beta = 3$. *(15 points)*
ii) For which values of $\alpha > 0$ and $\beta > 0$, the solution will be **C** = **B**? *(5 points)*

Solution: Let $BC = x$. The function to minimize is

$$f(x) = \alpha\sqrt{25 + x^2} + (12 - x)\beta, \quad x \in [0, 12].$$

Its derivative is

$$f'(x) = \frac{\alpha x}{\sqrt{25 + x^2}} - \beta.$$

In particular $f'(0) = -\beta < 0$, so the minimum never occurs at $x = 0$. This answers the second part.

When $\alpha = 5$, $\beta = 3$, the solution of $f'(x) = 0$ is at $15/4 \in [0, 12]$. To decide if this gives the minimum value or not we can do one of two things.

We either notice that

$$f''(x) = \frac{25\alpha}{(25 + x^2)^{3/2}} > 0, \text{ for } x \in [0, 12]$$

and conclude that $x = 15/4$ gives the minimum value,

or

we evaluate f at $x = 15/4$ and also at the end points

$$f(0) = 61, \ f(15/4) = 56, \ f(12) = 65,$$

and conclude that $x = 15/4 = 3.75$ gives the minimum.

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Q-4) Evaluate the integral

$$\int_0^{\pi/3} \sec^7 x\, dx,$$

assuming that $\int_0^{\pi/3} \sec^6 x\, dx \approx 25/3$ and $\int_0^{\pi/3} \sec^5 x\, dx \approx 21/4$.

Solution: We first try integration by parts by choosing $u = \sec^5 x$ and $dv = \sec^2 x\, dx$. This gives

$$\begin{aligned}
\int \sec^7 x\, dx &= \sec^5 x \tan x - 5 \int \sec^5 x \tan^2 x\, dx \\
&= \sec^5 x \tan x - 5 \int \sec^5 x (\sec^2 x - 1) x\, dx \\
&= \sec^5 x \tan x - 5 \int \sec^7 x\, dx + 5 \int \sec^5 x\, dx,
\end{aligned}$$

giving us

$$\int \sec^7 x\, dx = \frac{1}{6} \sec^5 x \tan x + \frac{5}{6} \int \sec^5 x\, dx.$$

Finally, evaluating this from 0 to $\pi/3$ we get

$$\int \sec^7 x\, dx = 13.618, \text{ or using the above approximation } \approx \frac{16}{\sqrt{3}} + \frac{35}{8} = 13.612.$$

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Q-5) Evaluate the integral

$$\int \frac{x}{x^2+2x+5}\,dx.$$

Solution:

$$\begin{aligned}
\frac{x}{x^2+2x+5} &= \frac{1}{2}\,\frac{(2x+2)-2}{x^2+2x+5} \\
&= \frac{1}{2}\,\frac{(2x+2)}{x^2+2x+5} - \frac{1}{x^2+2x+5} \\
&= \frac{1}{2}\,\frac{(2x+2)}{x^2+2x+5} - \frac{1}{4}\,\frac{1}{(\frac{x+1}{2})^2+1} \\
\frac{dx}{x^2+2x+5} &= \frac{1}{2}\,\frac{d(x^2+2x+5)}{x^2+2x+5} - \frac{1}{2}\,\frac{d(\frac{x+1}{2})}{(\frac{x+1}{2})^2+1} \\
\int \frac{x}{x^2+2x+5}\,dx &= \frac{1}{2}\,\ln(x^2+2x+5) - \frac{1}{2}\,\arctan(\frac{x+1}{2}) + C.
\end{aligned}$$