Date: 12 January 2012, Thursday
Time: 9:00-11:00
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 113 Calculus – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Every mathematical symbol and every equation you write must be part of a well constructed sentence. I will not read any hanging equations or symbols. I will not try to interpret your symbols. I will only grade what is written on your paper; I do not specialize in mind reading.**

NAME: STUDENT NO:

Q-1) Prove the Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then there is a point $c \in (a, b)$ such that

$$\int_a^b f(t)\,dt = f(c)(b-a). \quad \textit{(20 points)}$$

Give an example of an integrable function $f : [0, 1] \to \mathbb{R}$ which is not continuous on $[0, 1]$ but the above theorem still holds. (Prove your claim.) *(Extra 10 points)*
Give another example of an integrable function $g : [0, 1] \to \mathbb{R}$ which is not continuous on $[0, 1]$ and the above theorem does not hold. (Prove your claim.). *(Extra 10 points)*

Solution:

Since f is continuous on $[a, b]$ it has a maximum and minimum there. Let M and m be the maximum and minimum of f on $[a, b]$, respectively. Since

$$m \le f(x) \le M \ \text{ for } \ x \in [a, b],$$

we have after integrating

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

This gives after evaluating the left and right hand integrals

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Dividing both sides by $b - a$, we get

$$m \le \frac{\int_a^b f(x)\,dx}{b-a} \le M.$$

Since $\frac{\int_a^b f(x)\,dx}{b-a}$ is a value between m and M, by the intermediate value theorem there must exist a point $c \in (a, b)$ such that

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a},$$

which is the claim of the theorem.

Define two functions f and g on $[0, 1]$ as follows.

$$f(x) = \begin{cases} 1 & \text{if } x < 1/2, \\ 1/2 & \text{if } x = 1/2, \\ 0 & \text{if } x > 1/2. \end{cases}.$$

$$g(x) = \begin{cases} x & \text{if } x \neq 1/2, \\ 0 & \text{if } x = 1/2. \end{cases}.$$

Then

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = \frac{1}{2}.$$

We have $f(1/2) = 1/2$ but there is no point $c \in (0, 1)$ such that $g(c) = 1/2$. So the theorem holds for f but not for g.

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Q-2) Write your answers to the space provided. No partial credits.

- $f(x) = (\sin x)^x$, $f'(x) = (\sin x)^x(\ln \sin x + \frac{x \cos x}{\sin x})$.
- $f(x) = 7x^3 - 7^x$, $f'(x) = 21x^2 - 7^x \ln 7$.
- $f(x) = \arctan(\dfrac{1}{1+x^2})$, $f'(x) = \frac{1}{1+\left(\frac{1}{1+x^2}\right)^2} \frac{-2x}{(1+x^2)^2} = \frac{-2x}{x^4+2x^2+2}$.
- $f(x) = \displaystyle\int_{x^5}^{x^7} \sin t^3 \, dt$, $f'(x) = 7x^6 \sin x^{21} - 5x^4 \sin x^{15}$.

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Q-3) Write your answers to the space provided. No partial credits.

- $\displaystyle\int x\sin 5x\,dx = \frac{1}{25}\sin 5x - \frac{1}{5}x\cos 5x + C.$

- $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$

- $\displaystyle\int \frac{x}{(1+x)(2+x)}\,dx = -\ln(1+x) + 2\ln(2+x) + C.$

- $\displaystyle\int x\sqrt{(\pi+3x^2)}\,dx = \frac{1}{9}(\pi+3x^2)^{3/2} + C.$

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Q-4) Let $f(x) = x^3 - 9x^2 + 15x + 17$. Fill in the blanks below. Each correct answer is 3 points.

(i) $f'(x) = 3x^2 - 18x + 15$.

(ii) $f''(x) = 6x - 18$.

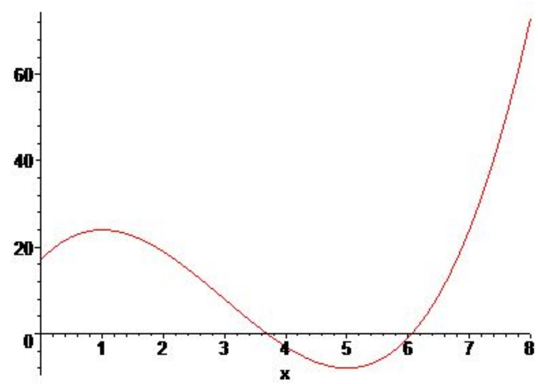
(iii) Critical points of f are $x = 1$ and $x = 5$

(iv) The inflection point of f is $x = 3$

(v) The maximum value of f on $[0, 8]$ occurs at $x = 8$

(vi) The minimum value of f on $[0, 8]$ occurs at $x = 5$

(vii) Sketch the graph of f on $[0, 8]$.



NAME: STUDENT NO:

Q-5) Let $D = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 + 2y = 1 \text{ and } y \geq 0\}$. Revolve the region D around x-axis and find the volume of the solid so obtained.

Solution:

From $x^2 + y^2 + 2y = 1$, we solve for y and find that $y = \sqrt{2 - x^2} - 1$. The volume of the solid of revolution is

$$\begin{aligned}
\pi \int_{-1}^{1} y^2 \, dx &= \pi \int_{-1}^{1} (3 - x^2 - 2\sqrt{2 - x^2}) \, dx \\
&= \pi \left(3x - \frac{1}{3}x^3 - x\sqrt{2 - x^2} - 2 \arcsin \frac{x}{\sqrt{2}} \Big|_{-1}^{1} \right) \\
&= \pi \left(\frac{10}{3} - \pi \right) \\
&\approx 0.6.
\end{aligned}$$