Due Date: November 30, 2011 Wednesday NAME:..

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Math 113 Calculus – Homework 2 – Solutions

1	2	TOTAL
75	25	100

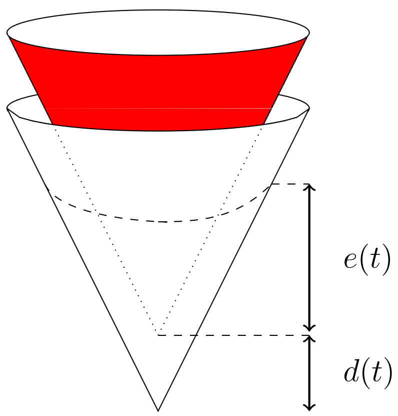
Please do not write anything inside the above boxes!

Check that there are 2 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. No hanging expressions will be read. No interpretation of your work will be attempted. Write everything you want us to know. Do not leave anything to mind reading; we do not do that in Math department, yet!

Q-1) We have a water tank in the form of an inverted cone containing some water. Another inverted cone of the same dimensions as the first one is touching the water level at the center of the water surface. Both cones are vertical. The second cone is moving down at a constant speed. How fast is the water level rising?

Hint: *To unify notation, for both of the cones let the ratio of base radius to height be $\alpha > 0$. Say the water level at time 0, when the second cone is just touching the water surface, is $d_0 > 0$. Say that the second cone is moving down with the constant speed of $s_0 > 0$ units per second. Let $h(t)$ denote the height of water at time t. Find $h'(t)$.*

Solution:



In the figure above $d(t)$ is the height of the tip the inner cone from base, and $e(t)$ is how far the inner cone is dipped into water.

At any time, the radius of the water surface is $\alpha h(t)$. At time $t = 0$, we have $h(0) = d_0$. Hence the volume of water in the tank is

$$V_0 = \frac{\pi\alpha^2}{3} d_0^3.$$

The second cone is moving down at the constant speed of s_0 units per second, and it has a distance of d_0 units to cover. So time is restricted by $0 \leq t \leq d_0/s_0$.

Let $h(t)$ denote the height of water at time t. When $t = d_0/s_0$, the two cones will coincide and will contain no volume between them. So $h(t)$ is defined for $0 \leq t < d_0/s_0$.

We observe that $h(t) = e(t) + d(t)$ for $t \in [0, d_0/s_0)$. Hence $h'(t) = e'(t) + d'(t)$.

Since the second cone is moving down with the constant speed of s_0 units per second, $d(t) = d_0 - s_0 t$.

It remains to determine $e(t)$. The volume $v(t)$ covered by the region of height $e(t)$ is

$$v(t) = V_0 - \frac{\pi\alpha^2}{3} d(t)^3 = \frac{\pi\alpha^2}{3}(d_0^3 - d(t)^3).$$

We can calculate this volume also geometrically.

$$v(t) = \frac{\pi\alpha^2}{3}\left((e(t) + d(t))^3 - d(t)^3 - e(t)^3\right).$$

Equating the two values of $v(t)$, we get

$$(3d(t))\, e(t)^2 + (3d(t))\, e(t) + \left(d(t)^3 - d_0^3\right) = 0. \tag{$*$}$$

Solving for $e(t) \geq 0$ we find

$$e(t) = -\frac{1}{2}d(t) + \sqrt{\frac{4d_0^3 - d(t)^3}{12d(t)}}. \tag{$**$}$$

Differentiate equation $(*)$ implicitly with respect to t and solve for $e'(t)$ to find

$$e'(t) = s_0 \frac{\left(e(t) + d(t)\right)^2}{d(t)^2 + 2d(t)e(t)}.$$

Finally, we have

$$h'(t) = e'(t) + d'(t) = s_0 \frac{\left(e(t) + d(t)\right)^2}{d(t)^2 + 2d(t)e(t)} - s_0, \text{ for } t \in [0, d_0/s_0),$$

where $d(t) = d_0 - s_0 t$ and $e(t)$ is as given in equation $(**)$.

Remark: Observe that $\lim\limits_{t \to (d_0/s_0)^-} h'(t) = \infty$, so at least theoretically, we exceed the speed of light!

NAME: STUDENT NO:

Q-2) For each positive integer n, calculate the limit $\lim\limits_{x\to 0} \dfrac{\cos^n x \sin^n x}{\cos x^n \sin x^n}$.

Solution: For any positive integer n, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos^n x \sin^n x}{\cos x^n \sin x^n} &= \lim_{x\to 0} \frac{\cos^n x}{\cos x^n} \lim_{x\to 0} \frac{\sin^n x}{\sin x^n} \\
&= \lim_{x\to 0} \frac{\sin^n x}{\sin x^n} \\
&= \lim_{x\to 0} \frac{n \sin^{n-1} x \cos x}{n x^{n-1} \cos x^n} \quad \text{(L'Hospital's Rule)} \\
&= \lim_{x\to 0} \frac{\sin^{n-1} x}{x^{n-1}} \lim_{x\to 0} \frac{\cos x}{\cos x^n} \\
&= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{n-1} \\
&= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^{n-1} \\
&= 1.
\end{aligned}$$

Solution 2: Here is an excellent solution form Emrah Karagöz who uses no L'Hospital!

He first writes

$$\frac{\cos^n x \sin^n x}{\cos x^n \sin x^n} = \frac{\cos^n x}{\cos x^n} \frac{\left(\dfrac{\sin x}{x}\right)^n}{\left(\dfrac{\sin x^n}{x^n}\right)}.$$

Then

$$\lim_{x\to 0} \frac{\cos^n x \sin^n x}{\cos x^n \sin x^n} = \lim_{x\to 0} \frac{\cos^n x}{\cos x^n} \frac{\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^n}{\lim\limits_{x\to 0} \left(\dfrac{\sin x^n}{x^n}\right)} = 1.$$