Date: 4 November 2011, Friday NAME:..
Time: 10:40-12:30
Ali Sinan Sertöz STUDENT NO:..

Math 113 Calculus – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Every mathematical symbol and every equation you write must be part of a well constructed sentence. I will not read any hanging equations or symbols. I will not try to interpret your symbols.**

NAME: STUDENT NO:

Q-1) Is $f(x) = \dfrac{1}{x}$ uniformly continuous on $(0, \infty)$? Prove your claim.

Solution:

f is not uniformly continuous here. We will assume uniform continuity and reach a contradiction.

Choose $\epsilon = 1$. Let $\delta > 0$ be the corresponding δ such that for any x, y in the domain $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$.

Choose x, y in the domain such that $|x - y| = \delta/2$. Then

$$|f(x) - f(y)| = \frac{|x - y|}{xy} = \frac{\delta}{2xy} < 1,$$

or equivalently

$$0 < \delta < 2xy \quad \text{for all } x, y \text{ in the domain with } |x - y| = \delta/2.$$

But this is clearly absurd since x and y can be chosen as small as we like and $2xy$ cannot always remain above a fixed δ.

We conclude that f is not uniformly continuous on $(0, \infty)$.

NAME: STUDENT NO:

Q-2) Is the following function differentiable at $x = 0$? Prove your claim.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution:

This function is differentiable at $x = 0$. We use the definition of derivative to show this. First observe that

$$\frac{f(h) - f(0)}{h} = h \sin \frac{1}{h},$$

and

$$0 \leq \left| \frac{f(h) - f(0)}{h} \right| = \left| h \sin \frac{1}{h} \right| \leq |h|$$

since $|\sin t| \leq 1$ for all t. But now using the squeeze theorem we conclude that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0,$$

and f is differentiable at $x = 0$ with $f'(0) = 0$.

NAME: STUDENT NO:

Q-3) Write your answers to the space provided. No partial credits.

- $f(x) = x^{\cos x}$, $f'(x) = x^{\cos x}(-\sin x \ln x + (\cos x)/x)$.

- $f(x) = x^5 - 7^x$, $f'(x) = 5x^4 - 7^x \ln 7$.

- $f(x) = \sin(\ln(\cos x))$, $f'(x) = \cos(\ln(\cos x))(1/\cos x)(-\sin x)$.

For the next two questions assume that $f(x) = x^2 + x + 1$ and $g(x) = \cos \pi x - \sin \pi x$.

- $(f \circ g)'(0) = -3\pi$.

- $(g \circ f)'(0) = \pi$.

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NAME: STUDENT NO:

Q-4) Let $f(x) = \arctan\left(\frac{x-1}{x+1}\right) - \arctan x$.

Find the domain of this function, *(5 points)*, and calculate explicitly $f(\sqrt{3})$, *(15 points)*.

Solution:

The domain is $\mathbb{R} - \{-1\}$. Check directly that $f'(x) = 0$ for all x in the domain, so the function is constant.

Hence $f(\sqrt{3}) = f(0) = \arctan(-1) = -\pi/4$.

NAME: STUDENT NO:

Q-5) Assume that the equation

$$e^{xy}\ln\frac{x}{y} - x - \frac{1}{y} = 0$$

defines y as a differentiable function of x. Find y' at the point $(e, \frac{1}{e})$.

Solution:

Differentiate implicitly with respect to x, keeping in mind that y is a function of x. You will get

$$e^{xy}(y + xy')\ln\frac{x}{y} + e^{xy}\,\frac{y}{x}\,\frac{y - xy'}{y^2} - 1 + \frac{y'}{y^2} = 0.$$

Now put $x = e$ and $y = 1/e$, solve for y' and find

$$y' = -e^{-2}.$$