Date: 30 December 2011, Friday
Time: 10:40-12:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 113 Calculus – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Every mathematical symbol and every equation you write must be part of a well constructed sentence. I will not read any hanging equations or symbols. I will not try to interpret your symbols. I will only grade what is written on your paper; I do not specialize in mind reading.**

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Q-1) Prove the Fundamental Theorem of Calculus: If f is continuous on $[0,1]$, then the function

$$F(x) = \int_0^x f(t)\,dt, \quad \text{for} \quad t \in [0,1]$$

is differentiable. *(20 points)*
Give an example of a function $f : [0,1] \to \mathbb{R}$ which is continuous on $[0,1]$ except at $x = 1/2$ and $F(x)$ is differentiable on $[0,1]$ except at $x = 1/2$. *(Extra 10 points)*
Give another example of a function $f : [0,1] \to \mathbb{R}$ which is continuous on $[0,1]$ except at $x = 1/2$ and $F(x)$ is nonetheless differentiable everywhere on $[0,1]$. *(Extra 10 points)*

Solution:

Fix a point $x_0 \in [0,1]$. We will show that $F'(x_0)$ exists. First observe that

$$\begin{aligned}
F(x_0+h) - F(x_0) &= \int_0^{x_0+h} f(t)\,dt - \int_0^{x_0} f(t)\,dt \\
&= \int_0^{x_0} f(t)\,dt + \int_{x_0}^{x_0+h} f(t)\,dt - \int_0^{x_0} f(t)\,dt \\
&= \int_{x_0}^{x_0+h} f(t)\,dt \\
&= f(c)h \quad \text{for some } c \text{ between } x_0 \text{ and } x_0+h \quad .
\end{aligned}$$

Then we have

$$\frac{F(x_0+h) - F(x_0)}{h} = f(c).$$

Since $c \to x_0$ as $h \to 0$, the limit of the above expression exists and is $f(x_0)$. Thus F is differentiable at every point of $[0,1]$.

The continuity of f is crucially needed at two places. First we needed it in using the mean value theorem for integrals. Then we needed it in finding the limit at the very end.

For the counterexamples asked in the problem, consider the functions

$$f(t) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 \\ 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

and

$$g(t) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2. \end{cases}$$

Then we have

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} 0 & \text{if } 0 \le x \le 1/2 \\ x - 1/2 & \text{if } 1/2 \le x \le 1, \end{cases}$$

and

$$G(x) = \int_0^x g(t)\,dt = x.$$

We see that F is not differentiable at $x = 1/2$, but G is differentiable everywhere.

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Q-2) Write your answers to the space provided. No partial credits.

- $f(x) = x^{\sin x}$, $f'(x) = x^{\sin x}(\cos x \ln x + (\sin x)/x)$.
- $f(x) = x^2 - 2^x$, $f'(x) = 2x - 2^x \ln 2$.
- $f(x) = \tan(\ln(\sec x))$, $f'(x) = \sec^2(\ln(\sec x))(\cos x)(\sec x \tan x)$.
- $f(x) = \displaystyle\int_{x^5}^{x^7} \sin t^3 \, dt$, $f'(x) = 7x^6 \sin x^{21} - 5x^4 \sin x^{15}$.

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Q-3) Write your answers to the space provided. No partial credits.

- $\int x \cos 3x \, dx = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C.$
- $\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C.$
- $\int \frac{1}{(1+x)(1+2x)} \, dx = - \ln|1+x| + \ln|1+2x| + C.$
- $\int x \sqrt{(1+x^2)} \, dx = \frac{1}{3}(1+x^2)^{3/2} + C.$

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Q-4) Evaluate the indefinite integral $\int (\cos x)\, e^{-x}\, dx$. *(10 points)*

Evaluate the indefinite integral $\int x(\cos x)\, e^{-x}\, dx$. *(5 points)*

Evaluate the improper integral $\int_{\pi}^{\infty} x(\cos x)\, e^{-x}\, dx$. *(5 points)*

Solution:

We try integration by-parts with the first integral by setting $u = \cos x$. This gives

$$\int \cos x\, e^{-x}\, dx = -\cos x e^{-x} - \int \sin x\, e^{-x}\, dx.$$

For the integral on the right hand side, we again try integration by-parts with $u = \sin x$ to obtain

$$\int \sin x\, e^{-x}\, dx = -\sin x e^{-x} + \int \cos x\, e^{-x}\, dx.$$

Solving for $\int \cos x\, e^{-x}\, dx$ and $\int \sin x\, e^{-x}\, dx$ from these equations, we get

$$\int \cos x\, e^{-x}\, dx = \frac{1}{2} e^{-x}(\sin x - \cos x) + C,$$

and

$$\int \sin x\, e^{-x}\, dx = -\frac{1}{2} e^{-x}(\sin x + \cos x) + C.$$

For the second integral, we try integration by-parts with $u = x$ to obtain

$$\int x \cos x\, e^{-x}\, dx = \frac{1}{2} x e^{-x}(\sin x - \cos x) - \frac{1}{2} \int e^{-x}(\sin x - \cos x)\, dx.$$

But from the previous calculations, we already know how to evaluate this last integral. This gives

$$\int x \cos x\, e^{-x}\, dx = \frac{1}{2} x e^{-x}(\sin x - \cos x) + \frac{1}{2} e^{-x} \sin x + C.$$

Evaluating the right hand side from π to infinity gives

$$\int_{\pi}^{\infty} x \cos x\, e^{-x}\, dx = -\frac{1}{2} \pi e^{-\pi} \approx -0.0678.$$

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Q-5) Let L be a line in the plane passing through the point $(27, 64)$. Let $A = (0, a)$ be the y-intercept of L, and $B = (b, 0)$ the x-intercept, where we consider only the case when a and b are positive. Find the smallest value that $|AB|$ can have.

Solution:

Let θ be the smaller angle the line L makes with the x-axis.

Let $P = (27, 64)$, $Q = (27, 0)$ and $R = (0, 64)$.

In ΔPQB, $|PB| = \dfrac{64}{\sin\theta}$.

In ΔPAR, $|PA| = \dfrac{27}{\cos\theta}$.

Hence $|AB| = f(\theta) = \dfrac{27}{\cos\theta} + \dfrac{64}{\sin\theta}$, for $\theta \in (0, \pi/2)$.

To minimize f we first find the roots of its derivative.

$$f'(\theta) = \frac{27\sin\theta}{\cos^2\theta} - \frac{64\cos\theta}{\sin^2\theta},$$

and

$$f'(\theta) = 0 \quad \text{when} \quad \tan^3\theta = \frac{64}{27}, \quad \text{or when} \quad \tan\theta = \frac{4}{3}.$$

Set $\theta_0 = \arctan\dfrac{4}{3}$. We check that

$$f(\theta_0) = 125, \ \lim_{\theta\to 0^+} f(\theta) = \infty, \quad \text{and} \quad \lim_{\theta\to(\pi/2)^-} f(\theta) = \infty.$$

We conclude that the smallest possible value of $|AB|$ is 125.