Date: 20 May 2005, Thursday

Q-1) Evaluate the following line integral

$$\int_C \frac{x^2}{1+y}\,dx + e^{xy}\,x\,dy$$

where C is the curve $y = x^2$ from the point $A(0,0)$ to the point $B(1,1)$.

Solution. Call the given line integral I. So

$$I = \int_C \frac{x^2}{1+y}\,dx + e^{xy}\,x\,dy.$$

Parameterize C as $C: \vec{\mathbf{r}}(t) = t\vec{\mathbf{i}} + t^2\vec{\mathbf{j}},\ 0 \le t \le 1$. Substituting $x = t,\ y = t^2$ into I we get

$$\begin{aligned} I &= \int_0^1 \left(\frac{t^2}{1+t^2} + e^{t\cdot t^2} t \cdot 2t\right)\,dt = \int_0^1 \left(\frac{t^2}{1+t^2} + 2e^{t^3}t^2\right)\,dt \\ &= \int_0^1 \left(1 - \frac{1}{1+t^2} + \frac{2}{3}\,e^{t^3}3t^2\right)\,dt = \left(t - \arctan t + \frac{2}{3}\,e^{t^3}\right)\Bigg|_0^1 \\ &= 1 - \underbrace{\arctan 1}_{\frac{\pi}{4}} + \frac{2}{3}\,e - 0 - \underbrace{\arctan 0}_{0} - \frac{2}{3}\underbrace{e^0}_{1} = 1 - \frac{\pi}{4} + \frac{2}{3}\,e - \frac{2}{3} = \frac{4+8e-3\pi}{12} \end{aligned}$$

Q-2) Evaluate the line integral

$$\int_C 2\,\cos y\,dx + \left(\frac{1}{y} - 2x\,\sin y\right)\,dy + \frac{1}{z}\,dz$$

where C is the curve of intersection of the surfaces $(8-\pi)x + 2y - 4z = 0$ and $16z = (32-\pi^2)x^2 + 4y^2$ from the point $A(0,2,1)$ to the point $B(1,\pi/2,2)$.

Solution. Call the given line integral I. So

$$I = \int_C 2\,\cos y\,dx + \left(\frac{1}{y} - 2x\,\sin y\right)\,dy + \frac{1}{z}\,dz.$$

Let

$$M = 2\cos y,\quad N = \frac{1}{y} - 2x\sin y,\quad P = \frac{1}{z}.$$

Then $I = \int_C M\ dx + N\ dy + P\ dz$. First we show that the differential form $M\ dx + N\ dy + P\ dz$ is exact.

$$\begin{aligned} \frac{\partial M}{\partial y} &= -2\sin y \quad , \quad \frac{\partial N}{\partial x} = -2\sin y \text{ are equal,} \\ \frac{\partial M}{\partial z} &= 0 \quad , \quad \frac{\partial P}{\partial x} = 0 \text{ are equal,} \\ \frac{\partial N}{\partial z} &= 0 \quad , \quad \frac{\partial P}{\partial y} = 0 \text{ are equal.} \end{aligned}$$

So the differential form is exact, thus there is a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = M = 2\cos y,\ \frac{\partial f}{\partial y} = N = \frac{1}{y} - 2x\sin y,\ \frac{\partial f}{\partial z} = P = \frac{1}{z}.$$

Integrating the first equation with respect to x we get

$$f(x,y,z) = 2x\cos y + g(y,z).$$

Then

$$\frac{\partial f}{\partial x} = N \Rightarrow -2x\sin y + \frac{\partial g}{\partial y} = \frac{1}{y} - 2x\sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y}.$$

Integrating with respect to y we get $g(y,z) = \ln|y| + h(z)$. So $f(x,y,z) = 2x\cos y + \ln|y| + h(z)$ and

$$\frac{\partial f}{\partial z} = P \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln|z| + C.$$

So

$$f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C.$$

So we have that

$$\begin{aligned} I &= f(1,\pi/2,2) - f(0,2,1) \\ &= (2\underbrace{\cos(\pi/2)}_{0} + \ln|\pi/2| + \ln|2| + C) - (0 + \ln|2| + \underbrace{\ln|1|}_{0} + C) = \ln(\pi/2) \end{aligned}$$

Q-3-A) Let $\vec{\mathbf{F}}(x,y,z) = M(x,y,z)\,\vec{\mathbf{i}} + N(x,y,z)\,\vec{\mathbf{j}} + P(x,y,z)\,\vec{\mathbf{k}}$ be a vector field such that M, N and P have continuous second order partial derivatives. Show that div (curl $\vec{\mathbf{F}}$) $= 0$.

Solution.

$$\text{curl}\,\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \vec{\mathbf{i}}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \vec{\mathbf{j}}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \vec{\mathbf{k}}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Then

$$\begin{aligned} \text{div}\,(\text{curl}\,\vec{\mathbf{F}}) &= \nabla \cdot (\text{curl}\,\vec{\mathbf{F}}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\partial^2 P}{\partial x\,\partial y} - \frac{\partial^2 N}{\partial x\,\partial z} - \frac{\partial^2 P}{\partial y\,\partial x} + \frac{\partial^2 M}{\partial y\,\partial z} + \frac{\partial^2 N}{\partial z\,\partial x} - \frac{\partial^2 M}{\partial z\,\partial y} = 0 \end{aligned}$$

by the equality of the mixed partial derivatives.

Q-3-B) Is there a vector field $\vec{\mathbf{G}}$ such that

$$\text{curl}\,\vec{\mathbf{G}} = 5x\,\vec{\mathbf{i}} + 7y\,\vec{\mathbf{j}} - 2z\,\vec{\mathbf{k}}?$$

If there is, find $\vec{\mathbf{G}}$. If there is no such $\vec{\mathbf{G}}$, explain why.

Solution. If there is such a $\vec{\mathbf{G}}$ then by part A) we have that

$$\text{div}\,(\text{curl}\,\vec{\mathbf{G}}) = \text{div}\,(\text{curl}\,(5x\,\vec{\mathbf{i}} + 7y\,\vec{\mathbf{j}} - 2z\,\vec{\mathbf{k}})) = 0,$$

that is $5 + 7 - 2 = 0$. Since this is not true, there is no such $\vec{\mathbf{G}}$.

Q-4) By using the Stokes' theorem, evaluate

$$\int_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $\frac{x}{3} + \frac{z}{4} = 1$ traversed in the counterclockwise sense when viewed from high above the xy-plane.

Solution. Call the given line integral I, so

$$I = \int_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz.$$

Let $\vec{\mathbf{F}} = (y-z)\vec{\mathbf{i}} + (z-x)\vec{\mathbf{j}} + (x-y)\vec{\mathbf{k}}$. Then $I = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\ ds$ and by Stokes' theorem $I = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma$, where

$$S : \underbrace{\frac{x}{3} + \frac{z}{4} - 1}_{f(x,y,z)} = 0, \quad \vec{\mathbf{n}} = \mp \frac{\nabla f}{|\nabla f|} = \mp \frac{\frac{1}{3}\vec{\mathbf{i}} + \frac{1}{4}\vec{\mathbf{k}}}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \mp \left(\frac{4}{5}\,\vec{\mathbf{i}} + \frac{3}{5}\,\vec{\mathbf{k}} \right).$$

Since $\vec{\mathbf{n}}$ has positive third component we have that

$$\vec{\mathbf{n}} = \frac{4}{5}\,\vec{\mathbf{i}} + \frac{3}{5}\,\vec{\mathbf{k}}.$$

Also

$$\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = \vec{\mathbf{i}}(-1-1) - \vec{\mathbf{j}}(1+1) + \vec{\mathbf{k}}(-1-1) = -2\vec{\mathbf{i}} - 2\vec{\mathbf{j}} - 2\vec{\mathbf{k}}.$$

So $I = \iint_S \left(-\frac{8}{5} - \frac{6}{5}\right) d\sigma = -\frac{14}{5} \iint_S d\sigma$. Taking the xy-plane as the ground plane we have $\vec{\mathbf{p}} = \vec{\mathbf{k}}$. Also the projection of S on the ground plane is the disk $R : x^2 + y^2 \leq 1$. Then

$$\begin{aligned} I &= -\frac{14}{5} \iint_S d\sigma = -\frac{14}{5} \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{\mathbf{p}}|}\, dA = -\frac{14}{5} \iint_R \frac{|\frac{1}{3}\vec{\mathbf{i}} + \frac{1}{4}\vec{\mathbf{k}}|}{|\frac{1}{4}|}\, dxdy \\ &= -\frac{14}{5} \iint_R \frac{\sqrt{\frac{1}{9} + \frac{1}{16}}}{\frac{1}{4}}\, dxdy = -\frac{14}{3}\ \text{Area}(R) = -\frac{14}{3}\pi. \end{aligned}$$