Date: 5 March 2005, Saturday SECTION: Kocatepe / Sertöz

Math 114 Calculus – Midterm Exam I – Solutions

Q-1) Find the coefficient a_{102} of x^{102} in the Taylor series $\sin\left(2x+\frac{\pi}{4}\right)=\sum_{n=0}^{\infty} a_n x^n$.

Solution:

$$\begin{array}{ll}
f(x) = \sin(2x+\pi/4) & f(0) = 1/\sqrt{2} \\
f'(x) = 2\cos(2x+\pi/4) & f'(0) = 2/\sqrt{2} \\
f''(x) = -2^2\sin(2x+\pi/4) & f''(0) = -2^2/\sqrt{2} \\
f'''(x) = -2^3\cos(2x+\pi/4) & f'''(0) = -2^3/\sqrt{2} \\
f^{(4)}(x) = 2^4\sin(2x+\pi/4) & f^{(4)}(0) = 2^4/\sqrt{2} \\
\vdots & \vdots
\end{array}$$

From this we see that $f^{(n)}(0)=\epsilon\ 2^n/\sqrt{2}$ where $\epsilon=1,1,-1,-1$ depending on whether n is $0,1,2,3$ mod 4, respectively. Note that $102\equiv 2$ mod 4 so $\epsilon=-1$ for $n=102$. The Taylor coefficient a_{102} is then given as

$$a_{102}=-\frac{2^{102}}{102!\ \sqrt{2}}.$$

Q-2) Let α be a constant. Find the value of α if

$$I=\int_1^{+\infty}\left(\frac{2x^2+\alpha x+\alpha}{x(2x+\alpha)}-1\right)\,dx=1.$$

Solution:

$$\frac{2x^2+\alpha x+\alpha}{2x^2+\alpha x}-1=\frac{\alpha}{2x^2+\alpha x}=\frac{A}{x}+\frac{B}{2x+\alpha}.$$

Solving for A and B we find $2A+B=0$ and $\alpha(A-1)=0$. Thus we have two caes:

Case 1: $\alpha=0$. Then clearly the above integral is zero and hence cannot be equal to 1. So $\alpha\neq 0$ and the next case holds.

Case 2: $\alpha \neq 0$. Then $A = 1$ and $B = -2$. We then find that

$$\begin{aligned}
\int_1^b \left(\frac{2x^2 + \alpha x + \alpha}{x(2x+\alpha)} - 1 \right) dx &= \int_1^b \left(\frac{1}{x} - \frac{2}{2x+\alpha} \right) dx \\
&= \left(\ln|x| - \ln|2x+\alpha| \Big|_1^b \right) \\
&= \ln b - \ln(2b+\alpha) + \ln(2+\alpha) \\
&= \ln \frac{b}{2b+\alpha} + \ln(2+\alpha)
\end{aligned}$$

Now we have

$$\begin{aligned}
\int_1^\infty \left(\frac{2x^2 + \alpha x + \alpha}{x(2x+\alpha)} - 1 \right) dx &= \lim_{b\to\infty} \left(\ln \frac{b}{2b+\alpha} + \ln(2+\alpha) \right) \\
&= -\ln 2 + \ln(2+\alpha) \\
&= \ln(1 + \alpha/2) \\
&= 1
\end{aligned}$$

This forces $1 + \alpha/2 = e$ and hence $\alpha = 2(e-1)$.

Q-3) Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ and κ for the curve $\vec{\mathbf{r}}(t) = (12\cos t, 12\sin t, 5t)$ at the point corresponding to $t = \pi/2$.

Solution:

$\mathbf{v} = (-12\sin t, 12\cos t, 5)$,

$|\mathbf{v}| = 13$,

$\mathbf{T} = \mathbf{v}/|\mathbf{v}| = (-\frac{12}{13}\sin t, \frac{12}{13}\cos t, \frac{5}{13})$,

$\frac{d\mathbf{T}}{dt} = (-\frac{12}{13}\cos t, -\frac{12}{13}\sin t, 0)$,

$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{12}{13}$,

$\mathbf{N} = \frac{d\mathbf{T}}{dt} / \left|\frac{d\mathbf{T}}{dt}\right| = (-\cos t, -\sin t, 0)$.

$\kappa = \left|\frac{d\mathbf{T}}{dt}\right| \frac{1}{|\mathbf{v}|} = \frac{12}{169}$.

Now we put $t = \pi/2$ to obtain:

$\mathbf{T} = (-\frac{12}{13}, 0, \frac{5}{13})$,

$\mathbf{N} = (0, -1, 0)$,

$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (\frac{5}{13}, 0, \frac{12}{13})$.

Note that κ was independent of t so is always 12/169.

Q-4) Calculate the distance from the point $\mathbf{r} = (3, 4, 4)$ to the line passing through the points $\mathbf{p} = (1, 2, 3)$ and $\mathbf{q} = (4, 6, 15)$.

Solution:

There are numerous ways of finding the required distance d. Here is one way:

Let $u = \vec{pr} = r - p = (2, 2, 1)$, and $v = \vec{pq} = q - p = (3, 4, 12)$.

Then $d = |u - \mathbf{proj}_v u|$. Recalling that $\mathbf{proj}_v u = \frac{u \cdot v}{v \cdot v} v$, we calculate immediately that $d = \sqrt{5}$.
