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Math 114 Calculus – Midterm Exam II
Solutions

Q-1) Let $f(x,y) = 3x^2 - 2y^2 + 3y$.
a) Find the minimum, maximum and the saddle points of $f(x,y)$ on $\mathbb{R}^2$, if they exist.
b) Find the absolute minimum and absolute maximum values of $f(x,y)$ on D where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Solution a:

$$\frac{\partial f}{\partial x} = 6x = 0, \quad \frac{\partial f}{\partial y} = -4y + 3 = 0 \Rightarrow x = 0, y = \frac{3}{4}.$$

So $P_0\left(0, \frac{3}{4}\right)$ is the only critical point.

$$f_{xx} = 6, f_{xy} = 0, f_{yy} = -4 \Rightarrow f_{xx}f_{yy} - (f_{xy})^2 = -24 < 0.$$

So there is a saddle point at $P_0(0, \frac{3}{4})$. Since P_0 is the only critical point, this function has no maximum and no minimum.

Solution b: The only critical point P_0 of $f(x,y)$ is an interior point of D, but it is a saddle point. So absolute minimum and absolute maximum values of $f(x,y)$ on D will take place on the boundary of D.
Boundary: $x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2$. Put into f.

$$3 - 3y^2 - 2y^2 + 3y = -5y^2 + 3y + 3 = g(y), \ -1 \leq y \leq 1.$$

Taking derivative

$$g'(y) = -10y + 3 = 0 \Rightarrow y = 3/10 \Rightarrow x^2 = \frac{91}{100} \Rightarrow x = \mp\frac{\sqrt{91}}{10}.$$

Considering the end points $y = \mp 1$ of the interval $[-1, 1]$, we get four candidates for the absolute minimum and maximum:

$$P_1\left(\frac{\sqrt{91}}{10}, \frac{3}{10}\right), P_2\left(-\frac{\sqrt{91}}{10}, \frac{3}{10}\right), P_3(0,1), P_4(0,-1).$$

Calculating $f(x,y)$ at these points:

$$f(P_1) = f(P_2) = \frac{69}{20}, \ f(P_3) = 1, \ f(P_4) = -5$$

yields that the absolute maximum value of $f(x,y)$ is $\frac{69}{20}$, taken at the points $P_1(\sqrt{91}/10, 3/10)$ and $P_2(-\sqrt{91}/10, 3/10)$, and absolute minimum value of $f(x,y)$ is -5 taken at the point $P_4(0,-1)$.

Q-2-A) Let $\omega = f(x, y)$ where $x = s^2 + t$ and $y = 3s - 2t$. Moreover assume that:

$$\frac{\partial f}{\partial x}(1,3) = 0, \quad \frac{\partial f}{\partial x}(1,-2) = 3, \quad \frac{\partial f}{\partial x}(0,1) = 2, \quad \frac{\partial f}{\partial y}(1,3) = 2, \quad \frac{\partial f}{\partial y}(1,-2) = 5, \quad \frac{\partial f}{\partial y}(0,1) = 4.$$

Find ω_s and ω_t at the point $(s, t) = (0, 1)$.

Q-2-B) Find $\frac{\partial z}{\partial x}$ at the point $(x, y, z) = (1, 2, 3)$, where z is a function of x and y and we have

$$x^y + y^z + z^x = xyz + 6.$$

Solution A: By the Chain Rule

$$\begin{aligned} \omega_s &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} \cdot 2s + \frac{\partial f}{\partial y} \cdot 3 \\ \omega_t &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y}(-2) \end{aligned}$$

When $s = 0, t = 1$, we have $x = 1, y = -2$. So

$$\begin{aligned} \omega_s &= \frac{\partial f}{\partial x}(1,-2) \cdot 0 + \underbrace{\frac{\partial f}{\partial y}(1,-2)}_{5} \cdot 3 = 5 \cdot 3 = 15 \\ \omega_t &= \underbrace{\frac{\partial f}{\partial x}(1,-2)}_{3} \cdot 1 + \underbrace{\frac{\partial f}{\partial y}(1,-2)}_{5} \cdot (-2) = 3 - 10 = -7 \end{aligned}$$

Solution B: First let us find the partial derivative of $u = z^x$ separately.

$$u = z^x \Rightarrow \ln u = x \ \ln z \Rightarrow \frac{1}{u}\ \frac{\partial u}{\partial x} = \ln z + x\ \frac{1}{z}\ \frac{\partial z}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = z^x \ \ln z + z^x \ \frac{x}{z}\ \frac{\partial z}{\partial x}.$$

Differentiating $(x^y + y^z + z^x = xyz + 6)$ with respect to x

$$y\ x^{y-1} + y^z \ \ln y\ \frac{\partial z}{\partial x} + z^x \ \ln z + z^x \ \frac{x}{z}\ \frac{\partial z}{\partial x} = yz + xy\ \frac{\partial z}{\partial x}.$$

Substituting $x = 1, y = 2, z = 3$, we get

$$2 + 2^3 \ \ln 2\ \frac{\partial z}{\partial x} + 3\ \ln 3 + 3\ \frac{1}{3}\ \frac{\partial z}{\partial x} = 6 + 2\ \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{4 - 3\ln 3}{8\ln 2 - 1}.$$

Q-3) Evaluate $\displaystyle\int_{\pi/2}^{\pi}\int_{x}^{\pi}\frac{\sin x}{4-\sin^2 y}\,dy\,dx.$

Solution:

$$\begin{aligned}
\int_{\pi/2}^{\pi}\int_{x}^{\pi}\frac{\sin x}{4-\sin^2 y}\,dy\,dx &= \int_{\pi/2}^{\pi}\int_{\pi/2}^{y}\frac{\sin x}{4-\sin^2 y}\,dx\,dy \\
&= \int_{\pi/2}^{\pi}\left(\left.\frac{-\cos x}{4-\sin^2 y}\right|_{\pi/2}^{y}\right)dy \\
&= \int_{\pi/2}^{\pi}\frac{\cos y}{\sin^2 -4}\,dy \\
&= \int_{0}^{1}\frac{du}{u^2-4} \\
&= \frac{1}{4}\int_{0}^{1}\left(\frac{du}{u+2}-\frac{du}{u-2}\right) \\
&= \frac{1}{4}\left(\ln\left.\frac{|u+2|}{|u-2|}\right|_{0}^{1}\right) \\
&= \frac{1}{4}\ln 3.
\end{aligned}$$

Q-4) The volume of a solid is

$$\int_{0}^{2}\int_{0}^{\sqrt{2x-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}dz\,dy\,dx.$$

a) Describe this solid.
b) Find its volume.

Solution: This is the region cut from the sphere of radius 2 and center the origin by half the cylinder $(x-1)^2+y^2=1$ with $y\geq 0$. To evaluate the volume we redescribe the region in cylindrical coordinates and set up the integral accordingly. Due to symmetry, we calculate the volume of the part with $z\geq 0$ and multiply by 2.

$$\begin{aligned}
\int_{0}^{2}\int_{0}^{\sqrt{2x-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}dz\,dy\,dx &= 2\int_{0}^{\pi/2}\int_{0}^{2\cos\theta}\int_{0}^{\sqrt{4-r^2}}r\,dz\,dr\,d\theta \\
&= 2\int_{0}^{\pi/2}\int_{0}^{2\cos\theta}r\sqrt{4-r^2}\,dr\,d\theta \\
&= -\frac{2}{3}\int_{0}^{\pi/2}\left.\left(4-r^2\right)^{3/2}\right|_{0}^{2\cos\theta}d\theta \\
&= \frac{16}{3}\int_{0}^{\pi/2}(1-\sin^3\theta)\,d\theta \\
&= \frac{16}{3}\left(\theta-\frac{1}{3}\cos^3\theta+\cos\theta\Big|_{0}^{\pi/2}\right) \\
&= \frac{8\pi}{3}-\frac{32}{9}=4.82...
\end{aligned}$$