Due on March 20, 2006, Monday.

MATH 114 Homework 5 – Solutions

1: Let $f(x, y) = \sin \ln(x^2 + y^2)$ where $x = \cos\theta$ and $y = 4\sin\theta$. Find $\left.\dfrac{\partial f}{\partial \theta}\right|_{\theta=\pi/4}$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \left[\cos\left(\ln(x^2+y^2)\right)\ \frac{2x}{x^2+y^2}\right][-\sin\theta] + \left[\cos\left(\ln(x^2+y^2)\right)\ \frac{2y}{x^2+y^2}\right][4\cos\theta] \\
\left.\frac{\partial f}{\partial \theta}\right|_{\theta=\pi/4} &= \left[\cos\left(\ln\frac{17}{2}\right)\ \frac{4}{17\sqrt{2}}\right]\left[-\frac{1}{\sqrt{2}}\right] + \left[\cos\left(\ln\frac{17}{2}\right)\ \frac{16}{17\sqrt{2}}\right]\left[\frac{4}{\sqrt{2}}\right] \\
&= \cos\left(\ln\frac{17}{2}\right)\ \frac{30}{17} \approx -0.95
\end{aligned}$$

2: Let $x^2 - xy + yz^3 + x^2z^2 - 2xy^3 = 0$ define z as a function of x and y. Find the equation of the tangent plane to this surface at the point $(1, 1, 1)$.

Solution: Take implicit derivative of this equation with respect to x and y separately to obtain

$$\begin{aligned}
2x - y + 3yz^2z_x + 2x^2zz_x + 2xz^2 - 2y^3 &= 0 \\
-x + 3yz^2z_y + z^3 + 2x^2zz_y - 6xy^2 &= 0
\end{aligned}$$

from which we find that at the point $(1, 1, 1)$ we should have $z_x = -\dfrac{1}{5}$ and $z_y = \dfrac{6}{5}$. The equation of the tangent plane at that point is

$$z = 1 - \frac{1}{5}(x-1) + \frac{6}{5}(y-1) = -\frac{1}{5}\,x + \frac{6}{5}\,y.$$

3: Consider the equations $w = x^4+3x^2y+xy^2+y^3$, $x = s^2+t^2$, $y = \cos\left(\frac{5\pi}{t^2+1}\right)$, $s = u+2v$ and $t = 3u+4v$. Find $\left.\dfrac{\partial w}{\partial u}\right|_{(u,v)=(1,0)}$.

Solution: We first prepare a table of values:

function	**value at** $(u,v) = (1,0)$
$s = u + 2v$	1
$t = 3u + 4v$	3
$x = s^2 + t^2$	10
$y = \cos\left(\dfrac{5\pi}{t^2+1}\right)$	0
$w_x = 4x^3 + 6xy + xy^2 + y^2$	4000
$w_y = 3x^2 + 2xy + 3y^2$	300
$x_s = 2s$	2
$x_t = 2t$	6
$y_s = 0$	0
$y_t = -\sin\left(\dfrac{5\pi}{t^2+1}\right)\dfrac{-10t\pi}{(t^2+1)^2}$	$\dfrac{3\pi}{10}$
$s_u = 1$	1
$t_u = 3$	3

Putting these together in $w_u = w_x\,(x_s\,s_u + x_t\,t_u) + w_y\,(y_s\,s_u + y_t\,t_u)$
we obtain $w_u = 80000 + 270\pi \approx 80848.23$.

4: Find the directional derivative of $f(x,y,z) = 2x^2 + 3y^3 + 4z^4$ at the point $(1,2,3)$ in the direction of $(4,5,6)$.

Solution:
$\nabla f = (4x, 9y^2, 16z^3)$, $\nabla f(1,2,3) = (4, 36, 432)$.

$|(4,5,6)| = \sqrt{77}$, $\vec{u} = (4/\sqrt{77}, 5/\sqrt{77}, 6/\sqrt{77})$.

$$D_{\vec{u}}f(1,2,3) = \nabla f(1,2,3)\cdot\vec{u} = \frac{2788}{\sqrt{77}} \approx 317.7.$$

5: Assume that $f(x,y) = 0$ defines a plane curve. Show that the gradient ∇f is orthogonal to the tangent line of the curve at every point where the curve is smooth.

Solution: If the curve is smooth at a point, then there is a local parametrization for the curve in the form $(x,y) = (x(t), y(t))$ for some real variable t. Then we have $f(x(t), y(t)) = 0$. Taking derivative with respect to t, using chain rule, we find $f_x \cdot x' + f_y \cdot y' = \nabla f \cdot (x', y') = 0$. Since (x', y') is the tangent vector of the curve at that point, this shows that ∇f is orthogonal to the curve at every point where the curve is smooth.
