Due on April 3, 2006, Monday.

MATH 114 Homework 6 – Solutions

1: Let $f(x,y) = 8x^3 + y^3 + 6xy$. Find local min/max, global min/max and saddle points, if they exist, for this function.

Solution:

$f_x = 24x^2 + 6y = 0$ and $f_y = 3y^2 + 6x = 0$ gives the critical points as $(0,0)$ and $(-1/2, -1)$.

We now apply the second derivative test:

$f_{xx} = 48x,\ f_{xy} = 6,\ f_{yy} = 6y$, so $\Delta(x,y) = 288xy - 36$.
$\Delta(0,0) < 0$, so $(0,0)$ is a saddle point.
$\Delta(-1/2,-1) > 0,\ f_{xx}(-1/2,-1) < 0$, so this is a local maximum point.
f has no global min/max since f becomes $\pm\infty$ as $(x,y) \to (\pm\infty, 0)$.

2: Let $f(x,y) = xy + 2x - \ln(x^2 y)$ where $x, y > 0$. Find local min/max, global min/max and saddle points, if they exist, for this function.

Solution:

$f_x = y + 2 - \dfrac{2}{x} = 0,\ f_y = x - \dfrac{1}{y} = 0$ gives $(1/2, 2)$ as the only critical point.

$f_{xx} = \dfrac{2}{x^2},\ f_{xy} = 1,\ f_{yy} = \dfrac{1}{y^2},\ \Delta = \dfrac{2}{x^2 y^2} - 1.$

At the critical point, $f_{xx} > 0$ and $\Delta > 0$, so the critical point is a local minimum.

f becomes infinite as (x,y) approaches the x-axis or the y-axis, which are the boundaries of the domain of f. Therefore the critical point $(1/2, 2)$ gives the global minimum of f.

3: Let $f(x,y) = x^2 + kxy + y^2$ where $k \in \mathbb{R}$. Find local min/max, global min/max and saddle points, if they exist, for this function, for each value of k.

Solution:

$f_x = 2x + ky = 0,\ f_y = kx + 2y = 0$ gives $y\left(2 - \dfrac{k^2}{2}\right) = 0.$

If $y = 0$, then $(0,0)$ is the only critical point of f.

$f_{xx} = f_{yy} = 2,\ f_{xy} = k$ so $\Delta = 4 - k^2$.

If $|k| < 2$, then $(0,0)$ is a local minimum point, but since it is the only critical point, it gives the global minimum.

If $|k| > 2$, then $(0,0)$ is a saddle point.

If $k = 2$, then $f = (x+y)^2 \geq 0$ and has a global minimum along the line $y = -x$.

If $k = -2$, then $f = (x-y)^2 \geq 0$ and has a global minimum along the line $y = x$.

Finally, if $y \neq 0$, then $k = \pm 2$ which is already examined.

4: Find the distance from the surface $z = x^2 + y^2 + 10$ to the plane $x + 2y - z = 0$. (This means you will calculate the minimum distance $|p - q|$ where p is on the surface and q is on the plane.)

Solution:

First method: If a plane P is defined by the equation $Ax + By + Cz = D$, then the distance from a point $q \in \mathbb{R}^3$ to the plane P is given by the formula $|q \cdot \vec{n}|$, where $\vec{n} = (A, B, C)/\sqrt{A^2 + B^2 + C^2}$. This formula can be easily derived by drawing some figure!

By roughly sketching the graphs of $z = x^2 + y^2 + 10$ and $x + 2y - z = 0$ we first notice that they do not intersect. If we choose q from the surface $z = x^2 + y^2 + 10$ and let $\vec{n} = (1, 2, -1)/\sqrt{6}$, we see that $q \cdot \vec{n}$ is either always positive or always negative. This value is negative at $(0, 0, 10)$ on the surface so we can take the distance function as $-q \cdot \vec{n}$. Putting in $z = x^2 + y^2 + 10$ we find the function

$$f(x, y) = x^2 + y^2 + 10 - x - 2y, \quad (x, y) \in \mathbb{R}^2$$

as the function to minimize. (We will divide this by $\sqrt{6}$ later.)

$f_x = 2x - 1 = 0,\ f_y = 2y - 2 = 0$ gives $(1/2, 1)$ as the only critical point.

$f_{xx} = f_{yy} = 2,\ f_{xy} = 0$, so $\Delta = 4 > 0$ and the critical point is a local minimum. But it is the only critical point, so it gives the global minimum.

The minimal distance is then calculated as $f(1/2, 1)/\sqrt{6} = \dfrac{35}{4\sqrt{6}}$.

Second method: Using the above analysis a little(!) we decide to minimize the function

$f(x,y,z) = \frac{1}{\sqrt{6}}(z - x - 2y)$ subject to the condition $g(x,y,z) = x^2 + y^2 + 10 - z = 0$.

We use Lagrange's method.

$\nabla f = \frac{1}{\sqrt{6}}(-1,-2,1)$, $\nabla g = (2x, 2y, -1)$.

From $\nabla f = \lambda \nabla g$ we first solve for λ and then find the critical point $(1/2, 1, 45/4)$, which gives $f = \frac{35}{4\sqrt{6}}$. By calculating f at another point, we find that this value is minimum.

5: Consider the surface S given by $f(x,y,z) = 0$ and assume that $p_0 = (x_0, y_0, z_0)$ is on the surface with $\frac{\partial f}{\partial z}(p_0) \neq 0$.
(i) Write the equation of the tangent plane to the surface S at p_0. From the equation of the tangent plane solve for z. Geometrically this is the linear approximation for the surface at the point p_0.
(ii) Now consider z as a function of the two independent variables x and y, say $z = g(x,y)$ with $z_0 = g(x_0, y_0)$. Assume as above that $f(x,y,g(x,y)) = 0$. Write a linear approximation for g at (x_0, y_0). i.e. write

$$L(x,y) = g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)\,(x - x_0) + \frac{\partial g}{\partial y}(x_0,y_0)\,(y - y_0).$$

Algebraically this is the linear approximation of the surface at the point p_0. How does this compare to the one found in the previous part? (This means you must calculate the partial derivatives of g in terms of the partial derivatives of f at the point p_0.)

Solution:

(i): The equation of the tangent plane to S at p_0 is given by

$$f_x(p_0)(x - x_0) + f_y(p_0)(y - y_0) + f_z(p_0)(z - z_0) = 0$$

from which we solve for z to find

$$z = z_0 - \frac{f_x(p_0)}{f_z(p_0)}(x - x_0) - \frac{f_y(p_0)}{f_z(p_0)}(y - y_0).$$

(ii): Differentiating both sides of the equation $f(x,y,g(x,y)) = 0$ with respect to x and y respectively we find

$$\begin{aligned} f_x(p_0) + f_z(p_0) g_x(x_0,y_0) &= 0 \\ f_y(p_0) + f_z(p_0) g_y(x_0,y_0) &= 0 \end{aligned}$$

which we solve to find

$$g_x(x_0, y_0) = \frac{f_x(p_0)}{f_z(p_0)} \text{ and } g_y(x_0, y_0) = \frac{f_y(p_0)}{f_z(p_0)}.$$

putting these into the given linear approximation, together with $z_0 = g(x_0, y_0)$, we find

$$L(x, y) = z_0 - \frac{f_x(p_0)}{f_z(p_0)}(x - x_0) - \frac{f_y(p_0)}{f_z(p_0)}(y - y_0)$$

which is precisely what we found in part (i).

Send comments and corrections to `sertoz@bilkent.edu.tr` please.
