Due on May 5, 2006, Friday, Class time. No late submissions!

MATH 114 Homework 8 – Solutions

1: Calculate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{F} = x\mathbf{i}+y^2\mathbf{j}+z^3\mathbf{k}$, and C is the helix $\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}$, $0\le t\le \pi$.

Solution: Let $\mathbf{r}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}$, $0\le t\le\pi$. Then

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi (\cos t, \sin^2 t, t^3)\cdot(-\sin t, \cos t, 1)\,dt \\
&= \int_0^\pi (-\cos t\sin t + \sin^2 t\cos t + t^3)\,dt \\
&= \left(-\frac{1}{2}\sin^2 t + \frac{1}{3}\sin^3 t + \frac{t^4}{4}\Big|_0^\pi\right) \\
&= \frac{\pi^4}{4}
\end{aligned}$$

2: Calculate $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, where $\mathbf{F} = (x^2-1)\mathbf{i}+(y+2)\mathbf{j}$, and C is that part of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ lying in the upper half plane and traversed clockwise.

Solution: Let $\mathbf{r}(t) = (-2\cos t, 3\sin t)$, $0\le t\le\pi$.
Then $(dx, dy) = (2\sin t, 3\cos t)\,dt$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_C \mathbf{F}\cdot(dy, -dx)\,dt \\
&= \int_0^\pi (4\cos^2 t - 1, 3\sin t + 2)\cdot(3\cos t, -2\sin t)\,dt \\
&= \int_0^\pi (12\cos^3 t - 3\cos t - 6\sin^2 t - 4\sin t)\,dt \\
&= \left(4\cos^2 t\sin t + 5\sin t + 3\sin t\cos t - 3t + 4\cos t\Big|_0^\pi\right) \\
&= -3\pi - 8.
\end{aligned}$$

3: Calculate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{F} = (e^x\cos y + yz)\mathbf{i} + (xz - e^x\sin y)\mathbf{j} + (xy+z)\mathbf{k}$, and C is the path $(\cos t + \sin^2 t)\ \mathbf{i} + (\sin t + 1 + \ln(1+\sin^2 t))\ \mathbf{j} + |\cos t|\sqrt{2-2\sin t}\ \mathbf{k}$, $0 \le t \le \pi$.

Solution: Here we check and find that $\mathbf{F}$ is conservative. In fact we can show that $\mathbf{F} = \nabla f$ where $f(x,y,z) = e^x\cos y + xyz + \frac{1}{2}z^2 + C$, for some constant $C \in \mathbb{R}$. The path starts at the point $(1,1,\sqrt{2})$ and ends at $(-1,1,\sqrt{2})$. The value of the integral is then $f(-1,1,\sqrt{2}) - f(1,1,\sqrt{2}) = -2\sinh 1\cos 1 - 2\sqrt{2} \approx -4.098$.

4: Calculate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, where $\mathbf{F} = \arctan(y/x)\ \mathbf{i} + \ln(x^2+y^2)\ \mathbf{j}$, and C is the positively oriented boundary of the region described in polar coordinates by the inequalities $1 \le r \le 2$ and $0 \le \theta \le \pi/2$.

Solution: Let R denote the above region. Using divergence theorem in the plane, the above integral is equal to

$$\int\!\!\int_R \frac{y}{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_1^2 \sin\theta\,dr\,d\theta = 0.$$

5: Find the area of the closed figure parameterized by $\cos^3\theta\ \mathbf{i} + \sin^3\theta\ \mathbf{j}$, $0 \le \theta \le 2\pi$.

Solution: We use the line integral area formula Area $= \displaystyle\frac{1}{2}\int_C (xdy - ydx)$. Here $x = \cos^3 t$ and $y = \sin^3 t$. Putting these in and simplifying gives

$$\text{Area} = \frac{3}{8}\int_0^{2\pi}\sin^2 2t\,dt = \frac{3}{8}\left(-\frac{1}{8}\sin 4t + \frac{t}{2}\Big|_0^{2\pi}\right) = \frac{3\pi}{8}.$$

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