Date: May 2006
Instructor: Ali Sinan Sertöz

Math 114 Calculus – Make-up Exam – Solutions

Q-1) Consider the power series $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$.

(i): Find its radius of convergence. (6 points)
(ii): Check convergence at the end points. (7 points each)

Solution: Applying the ratio test $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = |x| < 1$ for convergence gives the radius of convergence as 1.

When $x = -1$, the series converges by alternating series test.

When $x = 1$, the series converges by integral test.

Q-2) Assume that $w = f(u, v)$ satisfies $w_{uu} + w_{vv} = 0$.
Letting $u = \dfrac{x^2 - y^2}{2}$ and $v = xy$, calculate $w_{xx} + w_{yy}$.

Solution: This is an exercise in chain rule. Your calculations should give you $w_{xx} + w_{yy} = (x^2 + y^2)(w_{uu} + w_{vv}) = 0$.

Q-3) Write the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (3, -1, 2)$, where $f(x, y, z) = x^3 + 5xy^2z + 19y + xz^2 - 50$.
Write your answer in the form $Ax + By + Cz = D$.

Solution: The equation of this tangent is $\nabla f(3, -1, 2) \cdot (x - 3, y + 1, z - 2) = 0$. Simplifying this we get $41x - 41y + 27z = 218$.

Q-4) Evaluate $\displaystyle I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy.$

Solution:

$$I = \int_{-1}^{2} \int_{x^2}^{x+2} dydx = \int_{-1}^{2} \left(y\Big|_{x^2}^{x+2} \right) dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$\left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \Bigg|_{-1}^{2} \right) = \frac{9}{2}.$$

Q-5) Let C be the circle of intersection of the plane $4x-3y+5z=0$ with the sphere $x^2+y^2+z^2=11$, oriented counterclockwise when viewed from the north pole of the sphere.

Calculate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{F}=(2x-9y)\mathbf{i}+(5x-3z)\mathbf{j}+(y-6x)\mathbf{k}$ and $\mathbf{T}$ is the unit tangent vector of C with the given orientation.

Solution: Let D be the disc bounded by C with its unit normal vector $\mathbf{n}=(4,-3,5)/|(4,-3,5)|$. The area of D is 11π. The Stokes' theorem gives

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int\!\!\int_D \mathbf{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma \\
&= \frac{1}{\sqrt{50}}\int\!\!\int_D (4,6,14)\cdot(4,-3,5)\,d\sigma \\
&= \frac{68}{\sqrt{50}}\int\!\!\int_D d\sigma \\
&= \frac{68}{\sqrt{50}}\,11\pi = \frac{748\pi}{\sqrt{50}}.
\end{aligned}$$
