Date: 4 March 2006, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 114 Calculus – Midterm Exam I – Solutions

Q-1-a) Does the improper integral $\displaystyle\int_3^\infty \frac{e^{-x^2}}{(\ln x)^3}\,dx$ converge or diverge?

Solution: $\ln x > 1$ for $x \geq 3$, so $\displaystyle\frac{e^{-x^2}}{(\ln x)^3} < e^{-x^2} \leq e^{-x}$ for $x \geq 1$.

$\int_3^\infty e^{-x}\,dx = e^{-3} < \infty$, so the given integral converges by direct comparison.

Q-1-b) Find the value, if it exists, of the improper integral $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^k}$, where $k \geq 1$ is any real number.

Solution: Use the substitution $u = \ln x$ to write

$$\int_2^\infty \frac{dx}{x(\ln x)^k} = \int_{\ln 2}^\infty \frac{du}{u^k} = \begin{cases} \ln u \Big|_{\ln 2}^\infty = \infty & \text{if } k = 1, \\ \dfrac{1}{(1-k)u^{k-1}}\Big|_{\ln 2}^\infty = \dfrac{1}{(k-1)(\ln 2)^{k-1}} & \text{if } k > 1. \end{cases}$$

Q-2-a) Find $\displaystyle\lim_{n\to\infty}\left(\frac{7n+6}{7n+4}\right)^{5n}$, if it exists.

Solution: Let $A = \left(\dfrac{7n+6}{7n+4}\right)^{5n}$. Then

$$\lim_{n\to\infty} \ln A = 5 \lim_{n\to\infty} \frac{\ln\left(\frac{7n+6}{7n+4}\right)}{\frac{1}{n}}.$$

Now using L'Hopital's rule we get

$$\lim_{n\to\infty} \ln A = 5 \cdot \frac{7n+4}{7n+6} \cdot \frac{14n^2}{49n^2+56n+16} = \frac{10}{7}.$$

Hence $\displaystyle\lim_{n\to\infty}\left(\frac{7n+6}{7n+4}\right)^{5n} = e^{10/7}$.

We can also calculate this limit as follows: First let $m = 7n+4$. Then

$$\lim_{n\to\infty}\left(\frac{7n+6}{7n+4}\right)^{5n} = \lim_{m\to\infty}\left(\frac{m+2}{m}\right)^{5(\frac{m-4}{7})} = \lim_{m\to\infty}\left[\left(1+\frac{2}{m}\right)^m\right]^{\frac{5}{7}}\left[\left(1+\frac{2}{m}\right)^{-\frac{20}{7}}\right] = \left[e^2\right]^{\frac{5}{7}}[1] =$$
$e^{10/7}$.

Q-2-b) Does the series $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ converge or diverge?

Solution: Observe that $1+\frac{1}{2}+\cdots+\frac{1}{n} < 1+\ln n \leq n$ for all $n \geq 1$, where we write the first inequality by examining the graph of $y = 1/x$ and the second inequality is obvious if you consider the function $f(x) = x - 1 - \ln x$ for $x \geq 1$.

An easier observation is that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+1+1+\cdots+1 = n$ for $n > 1$.

Now if we let $a_n = \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$, we see that $a_n \geq \frac{1}{n}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ diverges by direct comparison with the harmonic series.

Q-3) Find all values of x for which the power series $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$ converges.

Solution: First let $a_n(x) = \frac{n^n}{n!} x^n$ and use the ratio test.

$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \left(1+\frac{1}{n}\right)^n |x| \to e\,|x|$ as $n \to \infty$. So the series converges absolutely for $|x| < 1/e$.

To check the end points we may use Stirling's formula, see page 759 exercise 90 and page 640 exercise 50.

As a consequence of Stirling's formula, for large n we have, $n! = \left(\frac{n+1}{e}\right)^{n+1} \sqrt{\frac{2\pi}{n+1}}\,(1+\epsilon(n))$ where $\lim_{n\to\infty} \epsilon(n) = 0$.

Hence $|a_n(\pm 1/e)| = \frac{n^n}{n!e^n} = \frac{n^n}{(n+1)^n} \cdot \frac{e}{\sqrt{2\pi}} \cdot \frac{1}{(n+1)^{1/2}} \cdot \frac{1}{1+\epsilon(n)} \to 0$ as $n \to \infty$.

Also observe that $\left|\frac{a_{n+1}(\frac{\pm 1}{e})}{a_n(\frac{\pm 1}{e})}\right| = \frac{\left(1+\frac{1}{n}\right)^n}{e} < 1$ for all $n \geq 1$, so $\left|a_{n+1}(\frac{\pm 1}{e})\right| < \left|a_n(\frac{\pm 1}{e})\right|$ for all $n \geq 1$.

From these we first conclude that the series converges for $x = -\frac{1}{e}$ by the alternating series test.

For $x = \frac{1}{e}$, we limit compare the series with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which diverges by p-test, $p < 1$.

$$\frac{a_n(1/e)}{1/n^{1/2}} = \frac{e}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{1+\epsilon(n)} \to \frac{1}{\sqrt{2\pi}} \text{ as } n \to \infty.$$

So the series diverges for $x = \frac{1}{e}$, and the interval of convergence is $[-\frac{1}{e}, \frac{1}{e})$.

Q-4) Find a power series solution to the initial value problem

$$y' - y = \frac{x^2}{2}, \quad y(0) = 0.$$

Can you recognize the solution in terms of elementary functions?

Solution: Putting $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ we immediately find that $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, $a_3 = 1/3!$, and in general $(n+1)a_{n+1} - a_n = 0$ for $n > 2$. This gives $a_n = \frac{1}{n!}$ for $n > 2$.

Thus $y = \sum_{n=3}^{\infty} \frac{x^n}{n!} = e^x - \frac{x^2}{2} - x - 1.$

Q-5) Find $\lim_{x \to 0} \frac{(\sin x^2)(e^x - 1) - x^3}{(1 - \cos 2x)(e^{x^2} - 1)}$, if it exists.

Solution: We use the Taylor series of the functions involved to find

$$\lim_{x \to 0} \frac{(\sin x^2)(e^x - 1) - x^3}{(1 - \cos 2x)(e^{x^2} - 1)} = \lim_{x \to 0} \frac{\frac{1}{2}x^4 + \frac{1}{6}x^5 + \cdots}{2x^4 + \frac{1}{3}x^6 + \cdots} = \frac{1}{4}.$$
