Date: 8 April 2006, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 114 Calculus – Midterm Exam II – Solutions

Q-1) Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^4y^2}{7x^8+3y^4}$, if it exists.

Solution: Approach the limit along the parabolas $y=\lambda x^2$,

$$\lim_{(x,\lambda x^2)\to(0,0)} \frac{5x^4y^2}{7x^8+3y^4} = \lim_{(x,\lambda x^2)\to(0,0)} \frac{5x^4(\lambda x^2)^2}{7x^8+3(\lambda x^2)^4} = \frac{5\lambda^2}{7+3\lambda^4}$$

which shows that the limit depends on the path. Hence the limit does not exist.

Q-2) Let $f(x,y)=x^2\ln\big(x^2+\sec(e^y-1)\big)$, where $x=\cos t+\sin t$ and $y=t+\tan t$. Find the derivative of f with respect to t at the point $t=0$.

Solution: The required value is $f_x(x(0),y(0))x'(0)+f_y(x(0),y(0))y'(0)$. So we calculate all these values separately.

$$\begin{aligned}
x(0) &= 1,\\
y(0) &= 0,\\
x'(t) &= -\sin t+\cos t,\\
x'(0) &= 1,\\
y'(t) &= 1+\sec^2 t,\\
y'(0) &= 2,\\
f_x(x,y) &= 2x\ln\big(x^2+\sec(e^y-1)\big)+\frac{2x^3}{x^2+\sec(e^y-1)},\\
f_x(1,0) &= 2\ln 2+1,\\
f_y(x,y) &= \frac{x^2\sec(e^y-1)\tan(e^y-1)e^y}{x^2+\sec(e^y-1)},\\
f_y(1,0) &= 0.
\end{aligned}$$

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Putting these together we find the result as $2\ln 2+1$.

Q-3) Find, if they exist, the local/global minimum/maximum and saddle points of the function

$$f(x,y) = x^4 - 8x^2 + 3y^2 - 6y.$$

Solution: $f_x = 4x(x^2-4) = 0,\ f_y = 6(y-1) = 0$ gives $(0,1)$, $(2,1)$ and $(-2,1)$ as the critical points.

$f_{xx} = 12x^2 - 16,\ f_{yy} = 6,\ f_{xy} = 0$ gives $\Delta = 72x^2 - 96$.

At $(0,1)$, $\Delta < 0$, so it is a saddle point.

At $(\pm 2, 1)$, $\Delta > 0$ and $f_{xx} > 0$, so they are both local minimum.

The function does not go to $-\infty$, and $f(\pm 2, 1) = -19$, so this value is the global minimum values of f.

Q-4) Find, if they exist, the local/global minimum/maximum and saddle points of the function

$$f(x,y) = x^3 + y^3 - 9xy - 35.$$

Solution: From $f_x = 3x^2 - 9y = 0$, and $f_y = 3y^2 - 9x = 0$ we find $(0,0)$ and $(3,3)$ as the critical points.

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -9$ and $\Delta = 36xy - 81$.

At $(0,0)$, $\Delta < 0$, so this is a saddle point.

At $(3,3)$, $\Delta > 0$ and $f_{xx} > 0$, so this is a local minimum point. Since the function goes both to plus and minus infinity, there is no global minimum.

Q-5) Find, if they exist, the minimum/maximum values of the function

$$f(x,y,z) = \frac{1}{xyz},$$

where $x, y, z > 0$ and satisfy $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$.

Solution: We use Lagrange's method with constraint $g = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1 = 0$.

$\nabla f = \left(\dfrac{-1}{x^2yz}, \dfrac{-1}{xy^2z}, \dfrac{-1}{xyz^2} \right)$, and $\nabla g = \left(\dfrac{2x}{4}, \dfrac{2y}{9}, \dfrac{2z}{25} \right)$.

From $\nabla f = \lambda \nabla g$ we find that $y = (3/2)x,\ z = (5/2)x$. Putting these into the constraint $g = 0$ we find $x = 2/\sqrt{3}$ (recall that $x, y, z > 0$). This then gives $y = 3/\sqrt{3}$ and $z = 5/\sqrt{3}$. Calculating f at these points we find $f = \sqrt{3}/10$. Since f is unbounded in this domain, this value must give the global minimum.
