Date: May 29, 2008, Thursday  NAME:....................................................
Time: 09:00-11:30
Ali Sinan Sertöz  STUDENT NO:....................................................

**Math 114 Calculus – Final Exam – Solutions**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
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| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

**PLEASE READ:**
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail but do not exaggerate. A correct answer without proper reasoning may not get any credit. Similarly an unnecessarily long explanation may be taken as an insult to intelligence and may not receive full credit. Moderation is the key word.

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The following formulas are copied from the book without any explanation. Use them with your own interpretation and responsibility.

$$\int \sin^4\theta \, d\theta = \frac{3}{8}\theta - \frac{3}{8}\sin\theta\cos\theta - \frac{1}{4}\sin^3\theta\cos\theta + C.$$

$$\int \cos^4\theta \, d\theta = \frac{3}{8}\theta + \frac{3}{8}\sin\theta\cos\theta + \frac{1}{4}\sin\theta\cos^3\theta + C.$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \oint_C M dy - N dx = \iint\limits_R (M_x + N_y)\ dx\, dy.$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M dx + N dy = \iint\limits_R (N_x - M_y)\ dx\, dy.$$

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint\limits_D \nabla\cdot\mathbf{F}\, dV, \qquad \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma.$$

$$\nabla\times\mathbf{F} = \operatorname{curl}\mathbf{F} = (P_y - N_z, M_z - P_x, N_x - M_y)\,.$$

$$\nabla\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = M_x + N_y + P_z.$$

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**Q-1)** Use the Lagrange multipliers method to find the point on the plane $ax + by + cz = d$ closest to the origin, where $a, b, c, d$ are real numbers with $a^2 + b^2 + c^2 \neq 0$.

Using your result show that the shortest distance is $\dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

**Solution:**

The function to minimize is $f = x^2+y^2+z^2$ which is the square of the distance function and clearly has no finite maximum. The constraint is the plane equation $g = ax + by + cz - d$.

We need to solve simultaneously $\nabla f = \lambda \nabla g$ and $g = 0$.

Solving for $x, y, z$ from the first equation and substituting into the second gives the value of $\lambda$, which in turn solves the problem.

The closest points are $x = (ad)/(a^2+b^2+c^2)$, $y = (bd)/(a^2+b^2+c^2)$, $z = (cd)/(a^2+b^2+c^2)$. Putting these into $f$ gives the above formula for distance.

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**Q-2)** Evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$.

**Solution:**

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy = \int_0^2 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^2 e^{x^3}\, x^2\, dx = \frac{1}{3}(e^8 - 1) \approx 993.$$

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**Q-3)** $D$ is the region in the upper half space $z \geq 0$, bounded by the sphere $x^2+y^2+z^2=16$ and the cylinder $x^2+y^2=2y$. Evaluate the following integral:

$$\iiint\limits_D z\,dx\,dy\,dz.$$

**Solution:**

$$\begin{aligned}
\iiint\limits_D z\,dx\,dy\,dz &= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{\sqrt{16-r^2}} z\,dz\,r\,dr\,d\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^{2\sin\theta}(16-r^2)\,r\,dr\,d\theta \\
&= 16\int_0^{\pi}\sin^2\theta\,d\theta - 2\int_0^{\pi}\sin^4\theta\,d\theta \\
&= \frac{29}{4}\,\pi.
\end{aligned}$$

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**Q-4)** $C$ is the smooth, closed, positively oriented curve given by the parametrization $x = \sin^5\theta + 2\cos\theta$, $y = \cos^5\theta + 2\sin\theta$, where $0 \le \theta \le 2\pi$.

It is also given that the area of the region $R$ bounded by the curve $C$ is $\dfrac{497\pi}{128}$.

Let $\mathbf{F} = (y^4 + xy^2 + 3x^2y,\ x^3 + x^2y + 4xy^3)$ and
$\mathbf{G} = (x^3 - x^2y - 4xy^3 + 4x,\ xy^2 + 124y + y^4 - 3x^2y)$ be two vector fields on $\mathbb{R}^2$.
Evaluate the following line integrals

$$\text{(i)} \quad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds, \quad \text{and} \quad \text{(ii)} \quad \oint_C \mathbf{G}\cdot\mathbf{n}\,ds,$$

where $\mathbf{T}$ is the unit tangent vector of $C$, and $\mathbf{n}$ is the unit outward normal of $C$.

**Solution: (i)**

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (N_x - M_y)\,dx\,dy = 0.$$

**Solution: (ii)**

$$\oint_C \mathbf{G}\cdot\mathbf{n}\,ds = \iint_R (M_x + N_y)\,dx\,dy = \iint_R 128\,dx\,dy = 497\pi.$$

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**Q-5)** Let $S$ be the surface given by $x^2+y^2+z=9$ with $z \geq 0$, and let $\mathbf{n}$ be its unit outward normal vector.
Let $\mathbf{F} = (y^2z + 3x^2y^2 + 2y + z^3x, 2xyz + y^4 + 2x^3y + 3x, x + y + \ln(x^4 + y^4 + 1))$ be a vector field defined on $\mathbb{R}^3$.
Evaluate the following integral:

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

**Solution:**

Let $C$ be the boundary of $S$. Then $C$ is the circle in the $xy$-plane centered at the origin with radius 3 and oriented positively. Let $D$ be the disk in the $xy$-plane bounded by $C$. We then have the following relations:

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

In the last integral $\mathbf{n} = \mathbf{k}$ and $d\sigma = dx\, dy$ since $D$ lies in the $xy$-plane.

We check easily that $\nabla \times \mathbf{F} \cdot \mathbf{k} = N_x - M_y = 1$, thus the value of the last integral is the area of $D$, which is simply $9\pi$. Thus

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 9\pi.$$