Calculus 114 – Homework 1 – Solutions
February 27, 2008 Wednesday.

Kummer's Test: Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

(1) $\sum_{n=1}^{\infty} a_n$ is convergent if and only if there exist *(i)* a sequence of positive terms $\{p_n\}$, *(ii)* a positive number $c > 0$, and *(iii)* an index N, such that

$$p_n \left(\frac{a_n}{a_{n+1}} \right) - p_{n+1} \geq c, \text{ for all } n \geq N.$$

(2) $\sum_{n=1}^{\infty} a_n$ is divergent if and only if there exist a sequence of positive terms $\{p_n\}$ such that $\sum_{n=1}^{\infty} 1/p_n$ diverges and

$$p_n \left(\frac{a_n}{a_{n+1}} \right) - p_{n+1} \leq 0, \text{ for all } n \geq N.$$

Raabe's Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.

(1) If there is an index N and a number $L > 1$ such that

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{L}{n}, \text{ for all } n \geq N,$$

then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\frac{a_{n+1}}{a_n} \geq 1 - \frac{1}{n}$, for all $n \geq N$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Gauss' Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.
Suppose there exist an index N, a real number $s > 1$, and constants $M > 0$ and $L \in \mathbb{R}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{L}{n} + \frac{f(n)}{n^s}, \text{ for all } n \geq N,$$

where $|f(n)| \leq M$ for all $n \geq N$.

(1) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $L \leq 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Optional exercise: (do not hand in your solutions to Ali for this question.)

Q-0) Prove Kummer's test. Observe that each statement is an "if and only if" statement. Using Kummer's test prove the other tests, including the ratio test. You may need to use Bernoulli's inequality at some point. Also recall the obvious fact that a series with positive terms converges if and only if its sequence of partial sums is bounded.

In the following exercises you can use any test, including the ones mentioned here.

It is highly recommended that you first solve the routine problems from Thomas's Calculus, pages 746-786. Do not consider yourself *studied* until you solve at most half of the exercises there.

Q-1) Consider the series $\sum_{n=1}^{\infty} \frac{(n!)^\alpha}{(3n)!}$, where α is a real constant. Find all values of α for which the series converges.

Solution:

This is solved by a direct application of the ratio test.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, where

$$\rho = \begin{cases} \infty & \text{if } \alpha > 3, \\ \frac{1}{27} & \text{if } \alpha = 3, \\ 0 & \text{if } \alpha < 3. \end{cases}$$

Hence the series converges if and only if $\alpha \leq 3$.

Q-2) Find all values of $\alpha \in \mathbb{R}$ for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^{\alpha}.$$

After you solve this, you should be able to give a quick answer for the same question for the following series:

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 6 \cdot 11 \cdots (5n+1)}{3 \cdot 8 \cdot 13 \cdots (5n+3)} \right)^{\alpha}.$$

Solution:

For this question, first observe that the following test is equivalent to the Gauss' test:

Modifies Gauss' Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.
Suppose there exist an index N, a real number $s > 1$, and constants $M > 0$ and $L \in \mathbb{R}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{L}{n + k_0} + \frac{g(n)}{n^2} + \frac{f(n)}{n^s}, \text{ for all } n \geq N,$$

where $|f(n)|, |g(n)| \leq M$ for all $n \geq N$, and k_0 is any fixed real number.

(1) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $L \leq 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

The proof of the equivalence uses trivial identities such as $\frac{1}{n + k_0} = \frac{1}{n} \frac{n}{n + k_0}$, and is left as a trivial exercise.

Another observation we will need is an application of Taylor's theorem.

Let $\phi(x) = (1 - x)^\alpha$, where $0 \leq x \leq 1/2$ and α is a fixed positive constant.

Taylor's theorem tells us that

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{1}{2}\phi''(c)x^2$$

for some c with $0 < c < x \leq 1/2$. This gives:

$$(1-x)^\alpha = 1 - \alpha x + \frac{1}{2}\alpha(\alpha-1)(1-c)^{\alpha-2}x^2.$$

For any $\alpha > 0$, it can be shown that $(1-c)^{\alpha-2} < 4$.

Now back to our problem:

For $\alpha \leq 0$, the series clearly diverges since the general terms are ≥ 1. So consider only $\alpha > 0$.

Letting $x = \dfrac{1/2}{n+1}$ and using the above expansion of $(1-x)^\alpha$ we see that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1/2}{n+1}\right)^\alpha = 1 - \frac{\alpha/2}{n+1} + \frac{f(n)}{n^2},$$

where $f(n) = \dfrac{1}{8}\alpha(\alpha-1)(1-c)^{\alpha-2}\dfrac{n^2}{(n+1)^2}$ and clearly $|f(n)| < \alpha|\alpha-1|$.

Now by the Modified Gauss' Test, the series converges if $\alpha/2 > 1$, and diverges if $\alpha/2 \leq 1$.

Conclusion: The series converges if and only if $\alpha > 2$.

The experience of this solution tells us that the second series will converge if and only if $2\alpha/5 > 1$, or $\alpha > 5/2$.

Q-3) Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

Solution:

This series behaves like the harmonic series;

$$\lim_{n\to\infty} \frac{1/n^{1+1/n}}{1/n} = 1,$$

so the series converges by limit comparing with the harmonic series.

Q-4) Find all pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ for which the following series converges.

$$\sum_{n=3}^{\infty} \frac{1}{n^a (\ln n)^b}.$$

Solution: Let $c_{a,b} = \dfrac{1}{n^a (\ln n)^b}$.

If $a \geq 2$ and $b \geq 0$, then $c_{a,b} \leq \frac{1}{n^2}$, and the series converges by direct comparison.

For $a = 1$ and $b \geq 0$, by examining the improper integral $\displaystyle\int_3^\infty \frac{dx}{x\,(\ln x)^b}$, we conclude that the series converges if $b > 1$ and diverges if $b = 0, 1$.

When $a = 0$ and $b \geq 0$, we recall that $\displaystyle\lim_{x \to \infty} \frac{(\ln x)^b}{x} = 0$, so $(\ln x)^b < x$ for all large x. This means that for all large n, we must have $c_{0,b} > 1/n$, and the series diverges by direct comparison.

Summing up: The series converges if and only if

$$(a \geq 2, b \geq 0) \text{ or } (a = 1, b \geq 2.)$$

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