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Time: 10:35-12:35
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Math 114 Calculus – Midterm Exam I – Solutions

Q-1) Does the following improper integral converge or diverge?

$$\int_3^\infty \frac{\ln x}{x(x-1)(x-2)}\,dx.$$

Solution:

Let $f(x) = \dfrac{\ln x}{x(x-1)(x-2)}$ and $g(x) = \dfrac{1}{x^2}$.

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, so $0 < f(x) < g(x)$ for all large x. Since $\displaystyle\int_3^\infty g(x)\,dx$ converges, so does the original integral by direct comparison.

Q-2) Does the following series converge or diverge?

$$\sum_{n=1}^\infty \frac{n}{5^n}\,.$$

If it converges, find its exact value.

Solution: We observe that on one hand

$$x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

and on the other hand

$$x\left(\frac{1}{1-x}\right)' = x\left(\sum_{n=0}^\infty x^n\right)' = \sum_{n=1}^\infty nx^n.$$

So we have

$$\frac{x}{(1-x)^2} = \sum_{n=1}^\infty nx^n.$$

Putting in $x = 1/5$ gives the value of the infinite sum as $\dfrac{5}{16}$.

Q-3) Find a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

for the following initial value problem:

$$y'' - xy = 2, \quad y(0) = 0, \quad y'(0) = 0.$$

Find the interval of convergence for the resulting series.

Solution:

$$y'' - xy = 2a_2 + (2 \cdot 3a_3 - a_0) + \cdots + (n(n+1)a_{n+1} - a_{n-3}) + \cdots = 2.$$

From this and the initial values we find that $a_0 = 0$, $a_0 = 0$ and $a_2 = 1$.

In general we find that $a_{3n} = 0$, $a_{3n+1} = 0$ and $a_{3n+2} = \dfrac{2 \cdot 3^n (n!)}{(3n+2)!} \cdot$ for $n \geq 1$.

This gives the solution as

$$y(x) = 2 \sum_{n=0}^{\infty} \frac{3^n (n!)}{(3n+2)!} x^{3n+2}.$$

By the ratio test we see that this series converges absolutely for all $x \in \mathbb{R}$.

Q-4) Find the interval of convergence for the following power series:

$$\sum_{n=0}^{\infty} \frac{3^n \, (n!)^2}{(2n)!} \, x^n.$$

Solution:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{3(n+1)^2}{(2n+1)(2n+2)} \, |x| \to \frac{3}{4}|x| \quad \text{as } n \to \infty.$$

Therefore the series converges absolutely for $|x| < 4/3$.

Next we check the end points.

For $|x| = 4/3$, we see that

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2n+2}{2n+1} > 1,$$

so $|a_{n+1}| > |a_n|$ and hence the general term does not converge to zero. The series then diverges for $x = \pm 4/3$.

Hence the interval of convergence is $-4/3 < x < 4/3$.

Q-5) Find the interval of convergence for the series

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} (x-3)^n.$$

Solution:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(\ln n+1)(n)}{(\ln n)(n+1)} |x-3| \to |x-3| \text{ as } n \to \infty.$$

Therefore the series converges absolutely for $|x-3|<1$.

Next we check the end points.

For $x=4$, the series diverges since $\dfrac{\ln n}{n} > \dfrac{1}{n}$.

For $x=2$, the series converges by the alternating series test. Here check that $\dfrac{\ln x}{x}$ is a decreasing function going to zero as x goes to infinity.

Hence the interval of convergence is $2 \leq x < 4$.

Bonus:) Fix a positive integer m and a positive real number α. Consider the sequence defined by the relations:

$$a_0 = 1, \quad a_n = \sqrt[m]{\alpha \sqrt[m]{a_{n-1}}} \text{ for } n \geq 1.$$

Find all pairs $(m, \alpha) \in \mathbb{N}^+ \times \mathbb{R}^+$ for which the sequence a_n converges as n goes to infinity. For those pairs (m, α) for which the sequence converges, find the limit of the sequence.

Solution:

For $m=1$, the sequence is $a_n = \alpha^n$. It converges to 1 if $\alpha = 1$ and converges to zero if $\alpha < 1$. For other values of α the sequence diverges to infinity.

For $m>1$, the sequence is $a_n = \alpha^{\frac{1}{m}+\frac{1}{m^3}+\cdots+\frac{1}{m^{2n-1}}}$.

$$\lim_{n\to\infty}\left(\frac{1}{m}+\frac{1}{m^3}+\cdots+\frac{1}{m^{2n-1}}\right) = \frac{m}{m^2-1}, \text{ so } \lim_{n\to\infty} a_n = \alpha^{\frac{m}{m^2-1}} \text{ for all } \alpha > 0.$$

Please forward any comments or questions to `sertoz@bilkent.edu.tr`
