Date: 23 May 2011, Monday
Time: 15:30-17:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 114 Calculus II – Final Exam – Solutions

1	2	3	4	5	*Bonus*	TOTAL
20	20	20	20	20	*(20)*	100

Please do not write anything inside the above boxes!

Check that there are 5+1 questions on your exam booklet. **Write your name on top of every page.** Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Also note that** if you write down something which you don't believe yourself, the chances are that I will not believe it either.

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Q-1) Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}$ for $n = 0, 1, \ldots$. Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

Find the interval of convergence of this series.

(Finding the radius of convergence is 10 points, checking at the end points is another 10 points.)

Hint: You may assume the fact that $\dfrac{1}{n+3} < a_n < \dfrac{1}{\ln n}$ *for all* $n \geq 4$.

Solution:

$\dfrac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \dfrac{2n+3}{2n+4}|x| \to |x|$ as $n \to \infty$. So the series converges for all $|x| < 1$.

Now we check the end points.

When $x = 1$, the series becomes $\sum a_n$. But since $a_n > 1/(n+3)$, the series diverges by comparison.

When $x = -1$, we have an alternating series. Since $a_{n+1} = a_n \dfrac{2n+3}{2n+4} < a_n$, the sequence a_n strictly decreases. Since $a_n < 1/\ln n$, the sequence a_n decreases to zero. Hence by the alternating series test, the sequence $\sum (-1)^n a_n$ converges.

The interval of convergence is $[-1, 1)$.

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Q-2) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where
$\mathbf{F}(x,y,z) = (yz,\ xz,\ xy)$ and C is the path $\mathbf{r}(t) = (t\cos t,\ t+\sin t,\ t^2+(2-\pi)t)$ for $0 \le t \le \pi$.

Solution:

Observe that **F** is conservative and $\mathbf{F} = \nabla f$ where $f(x,y,z) = xyz$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(-\pi, \pi, 2\pi) - f(0,0,0) = -2\pi^3.$$

If you do not make this observation, then you will have to evaluate

$$\int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt,$$

which is

$$\begin{aligned}&\int_0^\pi (-t(-4\,t^2\cos(t) + t^3\sin(t) - 6\,t\cos(t) + 2\ \sin(t)t^2 + \\ &3\,t\pi\ \cos(t) - t^2\pi\ \sin(t) - 3\,t\sin(t)\cos(t) + t^2 - 2\,t^2\,(\cos(t))^2 \\ &-4\ \sin(t)\cos(t) + 2\,t - 4\,t\,(\cos(t))^2 + 2\ \sin(t)\,\pi\ \cos(t) - t\pi + 2\,t\,(\cos(t))^2\,\pi))\ dt\end{aligned}$$

which also gives $= -2\pi^3$.

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Q-3) Find the minimum and maximum values of $f(x, y) = x^2 + xy + y^2 - y$ on the triangular region bounded by the lines $x = 0$, $y = 0$ and $y = 2x + 2$.

Solution:

The triangular region has the vertices $(0, 0), (0, 2), (-1, 0)$.

$f_x = 2x + y$, $f_y = x + 2y - 1$. The only critical point is $\left(-\frac{1}{3}, \frac{2}{3}\right)$ and is in the region since $0 < \frac{2}{3} < 2(-\frac{1}{3}) + 2 = \frac{4}{3}$.

Restrict f to the line $y = 2x + 2$.
$g(x) = f(x, 2x + 2) = 7x^2 + 8x + 2$ for $-1 \leq x \leq 0$.
$g'(x) = 14x + 8 = 0$ when $x = -\frac{4}{7}$. This gives the point $\left(-\frac{4}{7}, \frac{6}{7}\right)$.

Restrict f to the line $y = 0$.
$g(x) = f(x, 0) = x^2$ for $-1 \leq x \leq 0$, has no interior critical points.

Restrict f to the line $x = 0$.
$g(y) = f(0, y) = y^2 - y$ for $0 \leq y \leq 2$.
$g'(y) = 2y - 1 = 0$, $y = \frac{1}{2}$. This gives the point $\left(0, \frac{1}{2}\right)$.

Now we evaluate f at all these critical points and the end points.
$f(-\frac{1}{3}, \frac{2}{3}) = -\frac{1}{3}$.

$f(-\frac{4}{7}, \frac{6}{7}) = -\frac{2}{7}$.

$f(0, \frac{1}{2}) = \frac{1}{4}$.

$f(0, 0) = 0$.

$f(0, 2) = 2$.

$f(-1, 0) = 1$.

We can now see that the minimum value of f is $f(-\frac{1}{3}, \frac{2}{3}) = -\frac{1}{3}$ and
the maximum value is $f(0, 2) = 2$.

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Q-4) Evaluate the integral

$$\int_0^5 \int_0^{(3/5)\sqrt{25-y^2}} xye^{81-18x^2+x^4}\,dx\,dy.$$

Solution:

$$\begin{aligned}
\int_0^5 \int_0^{(3/5)\sqrt{25-y^2}} xye^{81-18x^2+x^4}\,dx\,dy &= \int_0^3 \int_0^{(5/3)\sqrt{9-x^2}} xye^{81-18x^2+x^4}\,dy\,dx \\
&= \frac{25}{18}\int_0^3 xe^{81-18x^2+x^4}\left(y^2\Big|_0^{(3/5)\sqrt{9-x^2}} \right)\,dx \\
&= \frac{25}{18}\int_0^3 x(9-x^2)e^{81-18x^2+x^4}\,dx \\
&= \frac{25}{36}\int_0^9 ue^{u^2}\,du \\
&= \frac{25}{72}\left(e^{u^2}\Big|_0^9 \right) \\
&= \frac{25}{72}\left(e^{81}-1\right) \\
&\approx 5\times 10^{34}.
\end{aligned}$$

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Q-5) Let $R > 0$ and $\alpha > 0$ be fixed numbers. Calculate the volume that is bounded from above by the sphere $x^2 + y^2 + z^2 = R^2$ and from the sides by the cone $\alpha^2 z^2 = x^2 + y^2$ with $z \geq 0$.

Solution:

The best way to evaluate the volume is through spherical coordinates.

$$\begin{aligned} Volume &= \int_0^{2\pi} \int_0^{\arctan \alpha} \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{2\pi R^3 \left(\sqrt{1 + a^2} - 1 \right)}{3\sqrt{1 + a^2}}. \end{aligned}$$

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Bonus :-) Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}$ for $n = 0, 1, \ldots$. Prove the fact that $a_n < \dfrac{1}{\ln n}$ for all $n \geq 4$.

You may need to know that $a_4 < 0.25$, $1/\ln 4 > 0.73$ *and* $\ln 16 > 2.77$.

No partial credits for incomplete answers.

Solution:

Proof by induction. By the hint, we know that $a_4 < 1/\ln 4$.

Assume that the inequality holds for n.

$a_{n+1} = a_n \dfrac{2n+3}{2n+4} < \dfrac{1}{\ln n} \dfrac{2n+3}{2n+4}$. This is going to be less than $\dfrac{1}{\ln(n+1)}$ if and only if

$$\frac{\ln(n+1)}{2n+4} < \frac{\ln n}{2n+3} \quad \text{holds for all} \quad n \geq 4.$$

This in turn holds if and only if the function

$$f(x) = \frac{\ln x}{2x+3}, \quad x \geq 4$$

is decreasing. For this we check its derivative.

$$f'(x) = \frac{2 + (3/x) - 2\ln x}{(2x+3)^2}.$$

Let $h(x) = 2 + (3/x) - 2\ln x$ for $x \geq 4$. Using the hint we see that $h(4) < 0$. Moreover, since $h'(x) = -\dfrac{3+2x}{x^2} < 0$ for $x \geq 4$, $h(x)$ is strictly decreasing, so is always negative. This says that $f(x)$ is always strictly decreasing. And this completes the proof.