Due Date: March 28, 2011 Monday NAME:..

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Math 114 Calculus – Homework 2 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Consider the functions

$$f(x,y)=\begin{cases}\frac{x^3+y^4}{x^2+y^2} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0).\end{cases} \text{ and } g(x,y)=\begin{cases}\frac{xy^2}{x^2+y^2} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0).\end{cases}.$$

(i): Is f differentiable at the origin?
(ii): Is g differentiable at the origin?
(iii): Is $f+g$ differentiable at the origin?

Solution:

(i):

$$f_x(0,0)=\lim_{x,y\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x,y\to 0}1=1.$$

$$f_y(0,0)=\lim_{x,y\to 0}\frac{f(0,y)-f(0,0)}{x-0}=\lim_{x,y\to 0}y=0.$$

$$\lim_{x,y\to 0}\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}=\lim_{x,y\to 0}\frac{y^4-xy^2}{(x^2+y^2)^{3/2}}\neq 0.$$

Hence f is not differentiable at the origin.

(ii): Similar calculations as above show that $g_x(0,0)=g_y(0,0)=0$ and that g is not differentiable at the origin.

(iii): Let $h=f+g$. From (i) and (ii) we immediately know that $h_x(0,0)=1$ and $h_y(0,0)=0$.

$$\lim_{x,y\to 0}\frac{h(x,y)-h(0,0)-h_x(0,0)x-h_y(0,0)y}{\sqrt{x^2+y^2}}=\lim_{x,y\to 0}\frac{y^4}{(x^2+y^2)^{3/2}}=0.$$

Hence $f+g$ is differentiable at the origin.

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Q-2) By approximately what percentage will the value of $\dfrac{xy^2}{x^2+y^2}$ increase or decrease at the point $(1, 2)$ if x increases by 4% and y increases by 7%?

Solution: Let $w = \dfrac{xy^2}{x^2+y^2}$. We use the convention that $\dfrac{\Delta w}{w} \approx \dfrac{dw}{w}$. We therefore first calculate w_x and w_y.

$$w_x = \frac{y^2 - x^2}{x^2+y^2}\frac{w}{x}, \; w_y = \frac{2x^2}{x^2+y^2}\frac{w}{y}.$$

Therefore

$$dw = w_x dx + w_y dy, \; \frac{dw}{w} = \frac{y^2-x^2}{x^2+y^2}\frac{dx}{x} + \frac{2x^2}{x^2+y^2}\frac{dy}{y}.$$

Now putting in $(x, y) = (1, 2)$, $dx/x = 4/100$ and $dy/y = 7/100$, we find that

$$\frac{\Delta w}{w} \approx \frac{dw}{w} = 5.2\%.$$

Another way of doing is to assume that $x = x(t), y = y(t)$ with $x(0) = 1, y(0) = 2$ and $x'(0) = 0.04, y'(0) = 0.14$. Then write $w(t) = w(x(t), y(t))$ and calculate $w'(0)$ using chain rule which should give $w'(0) = 0.0416$. Then, the percentage increase is given by $w'(0)/w(0) = 0.052 = 5.2\%$.

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Q-3) Show that the equations

$$\begin{cases} xy^2 + zu + v^2 = 3 \\ x^3z + 2y - uv = 2 \\ xu + yv - xyz = 1 \end{cases}$$

can be solved for x, y, z as functions of u, v near the point $p_0 = (x, y, z, u, v) = (1, 1, 1, 1, 1)$ and find $\left(\dfrac{\partial z}{\partial u}\right)_v$ at $(u, v) = (1, 1)$.

Solution: Let us first fix a notation.

$$\begin{aligned} F &= xy^2 + zu + v^2 - 3 \\ G &= x^3z + 2y - uv - 2 \\ H &= xu + yv - xyz - 1 \end{aligned}$$

We use the Implicit Function Theorem. We therefore need to calculate the following

$$\frac{\partial(F, G, H)}{\partial(x, y, z)}(p_0) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}(p_0) = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 4 \neq 0.$$

Therefore the above system of equations can be solved for x, y, z in terms of u, v.

To calculate $\left(\dfrac{\partial z}{\partial u}\right)_v (p_0)$ we first need the following.

$$\frac{\partial(F, G, H)}{\partial(x, y, u)}(p_0) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial u} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial u} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial u} \end{vmatrix}(p_0) = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -4.$$

Finally, the Implicit Function Theorem gives

$$\left(\frac{\partial z}{\partial u}\right)_v (p_0) = -\frac{\frac{\partial(F,G,H)}{\partial(x,y,u)}(p_0)}{\frac{\partial(F,G,H)}{\partial(x,y,z)}(p_0)} = -\frac{-4}{4} = 1.$$

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The first edition of our textbook was printed in 1983. At that time it was "Single-Variable Calculus" and was authored by Robert Alexander Adams alone. On page 225 there was a starred problem, problem 33, which was notoriously involved and was the talk of the town. Here it is for your enjoyment. Refresh your calculus tools from last semester.

Q-4) You are in a tank (the military variety) moving down the y-axis toward the origin. At time $t = 0$ you are 4 km from the origin, and 10 min later you are 2 km from the origin. Your speed is decreasing; it is proportional to your distance from the origin. You know that an enemy tank is waiting somewhere on the positive x-axis, but there is a high wall along the curve $xy = 1$ (all distances in km) preventing you from seeing just where it is. How fast must your gun turret be capable of turning to maximize your chances of surviving the enemy?

Solution:

Let $p(t)$ denote your distance from the origin at time t. This means that at time t, your tank is on the point $(0, p(t))$.

We know that your speed is proportional to your distance from the origin. This means

$$p'(t) = Ap(t), \text{ or equivalently } \frac{p'(t)}{p(t)} = A$$

for some constant A, where t is in minutes. Integrating both sides, we find that

$$p(t) = ce^{kt}$$

for some constants c and k. Putting in the informatin that $p(0) = 4$ and $p(10) = 2$, we find that

$$c = 4 \text{ and } k = -\frac{\ln 2}{10}.$$

But we will solve the problem by taking arbitrary $c > 0$ and $k < 0$.

For ease of notation, let us use p to denote $p(t)$ from now on.

The equation of a line from $(0, p)$ with slope m is $y = mx + p$. To find the intersection of this line with the curve $xy = 1$, we solve these equations simultaneously.

$$y = mx + p \ \ \& \ \ y = \frac{1}{x} \ \Rightarrow mx^2 + px - 1 = 0.$$

For this line to be tangent to $xy = 1$, the two intersection points must coincide, or the discriminant of the quadratic must vanish.

$$\Delta = p^2 + 4m = 0, \text{ or } m = -\frac{p^2}{4}.$$

Let θ be the smaller angle the tangent line $y = mx + p$ makes with the y-axis. Then

$$m = \tan(\pi/2 + \theta) = -\cot\theta,$$

or in other words,

$$\tan\theta = \frac{4}{p^2}, \quad \text{or equivalently} \quad \theta = \arctan\frac{4}{p^2}.$$

We are interested in the largest value of θ'. So define a funtion

$$f(t) = \theta'(t) = -\frac{8c^2ke^{2kt}}{c^4e^{4kt} + 16}, \quad t \in \mathbb{R}.$$

Since we can also write f as

$$f(t) = -\frac{8c^2k}{c^4e^{2kt} + 16e^{-2kt}}$$

we see that $\lim\limits_{t\to\pm\infty} = 0$, and since $k < 0$, f is always positive. So if we find a single critical point, it will give the maximum value of f.

We find that

$$f'(t) = \frac{16c^2k^2e^{2kt}\left(c^4e^{4kt} - 16\right)}{\left(c^4e^{4kt} + 16\right)^2}$$

and $f'(t) = 0$ if and only if

$$\left(c^4e^{4kt} - 16\right) = 0, \text{ or } t = \frac{\ln\left(\frac{16}{c^4}\right)}{4k}.$$

We now know that at this critical value of time f must have a maximum value. Evaluating f at this value of t we find

$$f\left(\frac{\ln\left(\frac{16}{c^4}\right)}{4k}\right) = -k.$$

Finally, putting in the value of k we found at the beginning, we find that the maximum speed of the gun turret must be at least

$$\frac{\ln 2}{10} \approx 0.0693\ rad/min$$

to maximize your chances of surviving the encounter.

Moreover, note that at time $t = \frac{\ln\left(\frac{16}{c^4}\right)}{4k}$, your tank will be at the point $(0, 2)$.

This problem was much more fun back in 1983 because at that time we had to do all these calculations by hand and the neat solutions that emerged after horrendous calculations were thrilling bonuses. Surely, now I did all the above calculations using a software and I don't even remember where the difficulties of calculations lied!