Due Date: May 9, 2011 Monday NAME:..

Ali Sinan Sertöz STUDENT NO:..

Math 114 Calculus – Homework 4 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Find the following sums

$$\sum_{n=0}^{\infty} \frac{n^3}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \frac{n^3}{2^n}.$$

Solution:

Let $f(x) = 1 + x + x^2 + \cdots + x^n + \cdots$.

Then $g(x) = x(x(xf'(x))')' = x + 8x^2 + \cdots + n^3 x^n + \cdots = \dfrac{x(x^2+4x+1)}{(1-x)^4}$.

Thus we have

$$\sum_{n=0}^{\infty} \frac{n^3}{2^n} = g(1/2) = 26 \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \frac{n^3}{2^n} = g(-1/2) = \frac{2}{27}.$$

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Q-2) Find the following limit.

$$\lim_{x\to 0} \frac{x \sec x^3 \sin x^2 \arctan x^3 - x^6}{x^6 \cos x^2\, e^{x^3} \tan(x^4/2)}.$$

Solution:

$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$

$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \cdots$

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \cdots$

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots$

$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \cdots$

$x \sec x^3 \sin x^2 \arctan x^3 - x^6 = -\frac{1}{6}x^{10} + \frac{1}{6}x^{12} + \frac{1}{120}x^{14} - 1/36\, x^{16} + \frac{1217}{5040}x^{18} + \cdots$

$x^6 \cos x^2\, e^{x^3} \tan(x^4/2) = \frac{1}{2}x^{10} + \frac{1}{2}x^{13} - \frac{1}{4}x^{14} + \frac{1}{4}x^{16} - \frac{1}{4}x^{17} + \frac{1}{16}x^{18} + \frac{1}{12}x^{19} + \cdots$

Cancelling out x^{10} and putting in $x = 0$, we see that the limit is $\frac{(-1/6)}{(1/2)} = -\frac{1}{3}$.

NAME: STUDENT NO:

Q-3) Find the minimum and maximum values of $f(x, y, z) = xz - y^2$ on the ball $x^2 + y^2 + z^2 \leq 1$.

Solution:

First we find the critical points of f inside the domain. $\nabla f = (z, -2y, x) = (0, 0, 0)$ only at the origin which is inside the domain. There, the values of the function is zero.

Then we apply Lagrange multipliers method on the boundary. There the restraint is $g = x^2 + y^2 + z^2 - 1 = 0$. Solving for $\nabla f = \lambda \nabla g$ we find that the critical points for this method are $(0, \pm 1, 0)$, $(\pm\sqrt{1/2}, 0, \pm\sqrt{1/2})$. Evaluating f at these points also, we find that the minimum value of f is -1 and the maximum value is $1/2$.

NAME: STUDENT NO:

Q-4) Find the volume of the region that lies inside the cylinder $x^2 + (y - R)^2 = R^2$ and the sphere $x^2 + y^2 + z^2 = 4R^2$, where $R > 0$.

Solution:

Let D be the half-disk in the xy-plane given by $x^2 + (y - R)^2 = R^2$ and $x \geq 0$. The required volume is 4 times the volume cut in the first octant due to symmetry.

$$\begin{aligned}
Volume &= 4 \iint_D z\, dA \quad \text{where} \quad z \geq 0 \\
&= 4 \iint_D \sqrt{4R^2 - x^2 - y^2}\, dxdy \\
&= 4 \int_0^{\pi/2} \int_0^{2R\sin\theta} \sqrt{4R^2 - r^2}\, r\, drd\theta \\
&= 4 \int_0^{\pi/2} \left(-\frac{1}{3}(4R^2 - r^2)^{3/2} \Big|_0^{2R\sin\theta} \right) \\
&= \frac{32R^3}{3} \int_0^{\pi/2} (1 - \cos^3\theta)\, d\theta \\
&= \frac{32R^3}{3} \left(t - \frac{1}{3}\cos^2\theta \sin\theta - \frac{2}{3}\sin\theta \Big|_0^{\pi/2} \right) \\
&= \left(\frac{32R^3}{3} \right) \left(\frac{\pi}{2} - \frac{2}{3} \right).
\end{aligned}$$