Date: 24 May 2011, Tuesday NAME:..
Time: 10:00-12:00
Ali Sinan Sertöz STUDENT NO:..

Math 114 Calculus II – Make-Up Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. **Write your name on top of every page.** Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Also note that** if you write down something which you don't believe yourself, the chances are that I will not believe it either.

NAME: STUDENT NO:

Q-1) Check if the following series converge or diverge:

$$\textbf{a)}\ \sum_{n=1}^{\infty} \frac{n^n}{e^n\, n!} \quad \text{and} \quad \textbf{b)}\ \sum_{n=1}^{\infty} (-1)^n \frac{n^n}{e^n\, n!}.$$

Hint: Stirling's formula says $\lim\limits_{n\to\infty} \dfrac{n!}{\sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n} = 1.$

Solution:

Let $a_n = \dfrac{n^n}{e^n\, n!}$. Then $a_{n+1} = a_n \dfrac{\left(1+\frac{1}{n}\right)^n}{e} < a_n$, so a_n is strictly decreasing.

Rewrite Stirling's formula as $\lim\limits_{n\to\infty} \dfrac{\sqrt{2\pi n}\, n^n}{e^n\, n!} = 1$. Let $\epsilon = 1/2$. For this ϵ there exists an index N such that for every $n \geq N$, we have

$$\frac{1}{2} = 1 - \epsilon < \frac{\sqrt{2\pi n}\, n^n}{e^n\, n!} < 1 + \epsilon = \frac{3}{2}$$

or equivalently

$$\frac{1}{2\sqrt{2\pi}} \frac{1}{n^{1/2}} < a_n < \frac{3}{2\sqrt{2\pi}} \frac{1}{n^{1/2}}.$$

This shows that $\lim\limits_{n\to\infty} a_n = 0$. Hence

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{e^n\, n!}$$

converges by the alternating series test. But comparing a_n by $1/n^{1/2}$, we see that

$$\sum_{n=1}^{\infty} \frac{n^n}{e^n\, n!}$$

diverges.

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Q-2) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$ and C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ traversed counterclockwise, where we take $a > 0$ and $b > 0$.

Solution:

Let $F = (M, N)$. Observe that $M_y = N_x$. So if C' is a circle of radius R centered at the origin with $0 < R < \min\{a, b\}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

which follows from Green's theorem.

Now parametrizing C' as $\mathbf{r}(t) = (R \cos t, R \sin t)$ and substituting in, we find that $\mathbf{F} \cdot d\mathbf{r} = dt$. Hence the integral becomes 2π.

NAME: STUDENT NO:

Q-3) Write the equation of the tangent plane to the surface

$$x^2 + 3xyz + \ln\frac{z^2+1}{10} + \cos(xy\pi) = 20$$

at the point $(1, 2, 3)$.

Solution:

Let $f = x^2 + 3xyz + \ln\frac{z^2+1}{10} + \cos(xy\pi) - 20$.

$\nabla f = \left(2x + 3yz - y\pi\sin(xy\pi),\ 3xz - x\pi\sin(xy\pi),\ 3xy + \frac{2z}{z^2+1}\right)$.

$\nabla f(1, 2, 3) = \left(20, 9, \frac{33}{5}\right)$.

Equation of the plane is $\nabla f(1, 2, 3) \cdot (x - 1, y - 2, z - 3) = 0$, or

$$20x + 9y + \frac{33}{5}z = \frac{289}{5}$$

or

$$100x + 45y + 33z = 289.$$

NAME: STUDENT NO:

Q-4) Find the surface area of the part of the cone $2\sqrt{x^2+y^2} = z$, $z \geq 0$, that lies over the disc D where D is in the xy-plane with center at $(1,0)$ and radius 1.

Hint: $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot k|} dA = |\vec{r}_u \times \vec{r}_v| dudv.$

Solution:

Let $f(x,y,z) = 4x^2 + 4y^2 - z^2$. Then $|\nabla f| = 2\sqrt{5}\,z$, $|\nabla f \cdot k| = 2z$ and $d\sigma = \sqrt{5}\ dA$. Hence integrating this over D we get $\sqrt{5}\,Area(D) = \sqrt{5}\,\pi$.

If you parametrize the cone with $\vec{r}(u,v) = (u\cos v,\ u\sin v, 2u)$ with $-\pi/2 \leq v \leq \pi/2$ and $0 \leq u \leq 2\cos v$, then $d\sigma = |\vec{r}_u \times \vec{r}_v| dudv = \sqrt{5}\,u\,dudv$. We again get

$$\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos v} u\,dudv = \sqrt{5}\,\pi.$$

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Q-5) Let $a, b, c > 0$ and D be the ellipsoidal ball $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$. Evaluate the integral

$$\iiint_D z^2 \, dV.$$

Solution:

Change coordinates as $x = au$, $y = bv$ and $z = cw$. Let B be the ball $u^2 + v^2 + w^2 \leq 1$. Then

$$\iiint_D z^2 \, dV = abc \iiint_B (cw)^2 \, dV = abc^3 \iiint_B w^2 \, dV.$$

Changing to spherical coordinates, we have $w = \rho \cos\phi$, $dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$ and

$$\begin{aligned} \iiint_B w^2 \, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \sin\phi \cos^2\phi \, d\rho d\phi d\theta \\ &= (2\pi) \left(\frac{\rho^5}{5} \bigg|_0^1 \right) \left(-\frac{\cos^3\phi}{3} \bigg|_0^{\pi} \right) \\ &= (2\pi)(\frac{1}{5})(\frac{2}{3}) \\ &= \frac{4\pi}{15}. \end{aligned}$$

hence

$$\iiint_D z^2 \, dV = \frac{4\pi}{15} abc^3.$$