Date: 7 March 2011, Monday
Time: 15:40-17:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 114 Calculus – Midterm Exam 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Also note that** if you write down something which you don't believe yourself, the chances are that I will not believe it either. Don't waste your time by trying your luck. Instead take your time to think.

Use the following formulas at your own discretion.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \cdots$$

$$\int_0^{1/2} \frac{dx}{1+x^3} = 0.485402\ldots$$

$$\int_0^{1/2} \frac{dx}{1+x^4} = 0.493958\ldots$$

NAME: STUDENT NO:

Q-1) Find the interval of convergence of the series $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^{2011}}$.

Solution:

Let $a_n = \frac{x^n}{(\ln n)^{2011}}$. Use ratio test to find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{\ln n}{\ln(n+1)} \right)^{2011} |x| = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{2011} |x| = |x|.$$

So the series converges absolutely for $|x| < 1$, and diverges for $|x| > 1$.

We now check the end points.

When $x = -1$, the series converges by the Alternating Series Test.

When $x = 1$, use Cauchy Condensation Test. Set $b_n = 2^n a_{2^n} = \frac{1}{(\ln 2)^{2011}} \frac{2^n}{n^{2011}}$. Then use ratio test for $\sum_{n=2}^{\infty} b_n$. Since $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 2$, it diverges. It follows that our series also diverges.

Hence the interval of convergence is $[-1, 1)$.

This was Question-3 in Recitation-2.

NAME: STUDENT NO:

Q-2) Find $\lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(n+2011)}{2011^n}$.

Solution:

Set $c = 1/2011$ and $k = 2011$.

Let $a_n = c^n(n+1)\cdots(n+k)$. Check that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1+k}{n+1} c = c$. Hence $\sum_{n=0}^{\infty} a_n$ converges by the ratio test. This means that the general term goes to zero.

Hence $\lim_{n\to\infty} \frac{(n+1)(n+2)\cdots(n+2011)}{2011^n} = 0$.

We can also show this by L'Hopital's rule. $c^n(n+1)\cdots(n+k) = \frac{(n+1)\cdots(n+k)}{(1/c)^n}$. After k applications of L'Hopital we obtain

$$\lim_{n\to\infty} \frac{(n+1)\cdots(n+k)}{(1/c)^n} = \lim_{n\to\infty} \frac{k!}{[\ln(1/c)]^k(1/c)^n} = 0.$$

This was Question-5 in Recitation-2.

NAME: STUDENT NO:

Q-3) Prove the following part of the *Cauchy Condensation Test*:
Assume that $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \geq 0$.
Then, $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Solution:

First of all let us fix a notation. Let $s_n = a_1 + \cdots + a_n$ and $t_n = a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n}$.

Assume that $\lim_{n\to\infty} t_n = t$ and observe that since each $a_n \geq 0$, we have s_n as an increasing sequence. If we can show that it is bounded from above, we will conclude that it converges, which in turn will imply that $\sum_{n=0}^{\infty} a_n$ converges. For this consider the following inequalities.

$$
\begin{array}{rcl}
a_1 & = & a_1 \\
2a_2 & \geq & a_2 + a_3 \\
4a_4 & \geq & a_4 + a_5 + a_6 + a_7 \\
\cdots & & \\
2^n a_{2^n} & \geq & a_{2^n} + a_{2^n+1} + \cdots + a_{2^{n+1}-1}.
\end{array}
$$

Adding these side by side we get $t_n \geq s_{2^{n+1}-1}$. Since $t \geq t_n$ and since $s_{2^{n+1}-1}$ is an increasing sequence, it follows that being bounded from above it converges, say to a number c. So we have $s_{2^{n+1}-1} < c$.

For any n, $n \leq 2^{k+1} - 1$ for some k. (In fact $k \geq (\ln(n+1)/\ln 2) - 1$.) Since s_n is increasing, $s_n \leq s_{2^{k+1}-1} < c$. Hence, s_n is increasing and bounded from above and converges.

This was Question-1 in Recitation-1.

NAME: STUDENT NO:

Q-4) Find $\lim_{x\to 0} \dfrac{x^4 + (\tan x^2)(2-2\sec x)}{(\sin x^2)(2-2\cos x) - x^4}$.

Solution:

We use Taylor expansions of the numerator.

$$\begin{aligned} x^4 + \left(\tan x^2\right)(2-2\sec x) &= x^4 + \left(x^2 + \frac{x^6}{3} + \frac{2x^{10}}{15} + \cdots\right)\left(-x^2 - \frac{5x^4}{24} - \frac{61x^6}{720} - \cdots\right) \\ &= -\frac{5x^6}{12} - \frac{181x^8}{360} - \frac{93x^{10}}{448} - \cdots . \end{aligned}$$

And the denominator:

$$\begin{aligned} \left(\sin x^2\right)(2-2\cos x) - x^4 &= \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots\right)\left(x^2 - \frac{x^4}{12} + \frac{x^6}{360} - \cdots\right) - x^4 \\ &= -\frac{x^6}{12} - \frac{59x^8}{360} - \frac{31x^{10}}{2240} - \cdots . \end{aligned}$$

It then follows that

$$\begin{aligned} \lim_{x\to 0} \frac{x^4 + (\tan x^2)(2-2\sec x)}{(\sin x^2)(2-2\cos x) - x^4} &= \lim_{x\to 0} \frac{\left(-\frac{5x^6}{12} - \frac{181x^8}{360} - \frac{93x^{10}}{448} - \cdots\right)}{\left(-\frac{x^6}{12} - \frac{59x^8}{360} - \frac{31x^{10}}{2240} - \cdots\right)} \\ &= \lim_{x\to 0} \frac{\left(-\frac{5}{12} - \frac{181x^2}{360} - \frac{93x^4}{448} - \cdots\right)}{\left(-\frac{1}{12} - \frac{59x^2}{360} - \frac{31x^4}{2240} - \cdots\right)} \\ &= \frac{\left(-\frac{5}{12}\right)}{\left(-\frac{1}{12}\right)} \\ &= 5. \end{aligned}$$

A similar question was solved in Recitation-4.

NAME: STUDENT NO:

Q-5) Find the sum

$$S = 1 - \frac{1}{2^4\,4} + \frac{1}{2^7\,7} - \frac{1}{2^{10}\,10} + \cdots + \frac{(-1)^n}{2^{3n+1}\,(3n+1)} + \cdots$$

Solution:

Consider the function

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{(3n+1)} + \cdots\,, \text{ for } |x| < 1.$$

We then have $S = 1/2 + f(1/2)$.

We observe that

$$f'(x) = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots = \frac{1}{1+x^3}.$$

It then follows that

$$f(x) = \int_0^x \frac{dt}{1+t^3}, \text{ for } |x| < 1.$$

Finally

$$S = 1/2 + f(1/2) = 1/2 + \int_0^{1/2} \frac{dt}{1+t^3} = 0.5 + 0.485402\cdots = 0.985402\ldots.$$

This was Question-4 in Homework-1. A very similar question was solved in Recitation-4.