Date: 25 April 2011, Monday
Time: 15:40-17:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 114 Calculus – Midterm Exam 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Also note that** if you write down something which you don't believe yourself, the chances are that I will not believe it either. Don't waste your time by trying your luck. Instead take your time to think.

NAME: STUDENT NO:

Q-1) Find all unit vectors $\vec{u} = (a, b)$ such that the directional derivative of

$$f(x, y) = x^4y + y^3 + xy + 4x + 3y + 12$$

at the origin in the direction of $\vec{u}$ is 3.

Solution:

Since $\vec{u}$ is a unit vector, we must have $a^2 + b^2 = 1$.

The directional derivative of f at the origin in the direction of $\vec{u}$ is

$$\begin{aligned} D_{\vec{u}}f(0,0) &= \nabla f(0,0) \cdot \vec{u} \\ &= (4,3) \cdot (a,b) \\ &= 4a + 3b \\ &= 4a + 3\sqrt{1-a^2} \\ &= 3. \end{aligned}$$

Solving this for a gives $a = 0$ and $a = \frac{24}{25}$.

Possible candidates for $\vec{u}$ are $(0, \pm 1)$ and $(\frac{24}{25}, \pm\frac{7}{25})$. After calculating the directional derivatives along these directions, we find that the only two vectors satisfying the requirement are

$$\vec{u} = (0, 1), \text{ and } \vec{u} = (\frac{24}{25}, -\frac{7}{25}).$$

NAME: STUDENT NO:

Q-2) Find the minimum and maximum values of

$$f(x,y) = 27x + 2y + \frac{4}{xy}$$

in the first quadrant $x, y > 0$.

Solution:

As either of x and y approaches to zero, the function becomes unbounded because of the $\frac{4}{xy}$ term. Also, as either of x and y approaches to infinity, the function again becomes unbounded because of the $27x + 2y$ term. Therefore, there must be a minimum value of the function somewhere in the first quadrant $x, y > 0$.

To find the minimum value, we first find the critical points.

$f_x = 27 - \frac{4}{x^2 y} = 0$ and $f_y = 2 - \frac{4}{xy^2} = 0$.

Solving these we find that the only critical point is $(\frac{2}{9}, 3)$ which must give the minimum value.

Finally we find that the minimal value of f in the first quadrant is $f(\frac{2}{9}, 3) = 18$.

NAME: STUDENT NO:

Q-3) Find the points on the curve $17x^2 + 12xy + 8y^2 = 100$ that are closest to and farthest away from the origin.

Solution:

This is the equation of an ellipse, so there exist points minimizing and maximizing the square of the distance function. We can therefore use Lagrange multiplier method.

The function to use is $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 17x^2 + 12xy + 8y^2 - 100$.

The Lagrangian in this case is

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(17x^2 + 12xy + 8y^2 - 100).$$

Its critical points are given by $L_x = 0$, $L_y = 0$ and $L_\lambda = 0$.

Solving these equations, we find that the points closest to the origin on this ellipse are

$$(2, 1), \quad (-2, -1)$$

and the points farthest away from the origin are

$$(2, -4), \quad (-2, 4).$$

This is Example 2 in the book on page 759.

NAME: STUDENT NO:

Q-4) Find the volume of the region lying inside all three of the circular cylinders

$$x^2 + y^2 = a^2, \quad x^2 + z^2 = a^2 \text{ and } y^2 + z^2 = a^2$$

where $a > 0$.

Solution:

After a careful sketching of the solid under question, we find that the volume is

$$\begin{aligned} V &= 16 \left(\int_0^{a/\sqrt{2}} \int_0^x \sqrt{a^2 - x^2}\, dydx + \int_{a/\sqrt{2}}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dydx \right) \\ &= 16 \left(1 - \frac{1}{\sqrt{2}} \right) a^3. \end{aligned}$$

This is Exercise 25 of exercise group 14.4 on page 817. The answer is given at the back of the book.

NAME: STUDENT NO:

Q-5) Evaluate the integral

$$\int_C \vec{F} \cdot \, d\vec{r}$$

where C is the path from $(1, \sqrt{3})$ to $(2, 0)$ along the circle $x^2 + y^2 = 4$ and $\vec{F} = (xy, 2y^2)$.

Solution:

We first parameterize the path.

$\vec{r} = (x, y) = (2\cos\theta, 2\sin\theta)$, where θ runs from $\pi/3$ to 0.

$d\vec{r} = (-2\sin\theta, 2\cos\theta)\, d\theta$.

$\vec{F}(x, y) = (4\sin\theta\cos\theta, 8\sin^2\theta)$,

Finally

$$\begin{aligned}
\int_C \vec{F} \cdot \, d\vec{r} &= \int_{\pi/3}^{0} (4\sin\theta\cos\theta, 8\sin^2\theta) \cdot (-2\sin\theta, 2\cos\theta)\, d\theta \\
&= \int_{\pi/3}^{0} (8\sin^2\theta\cos\theta)\, d\theta \\
&= \left(\frac{8}{3}\sin^3\theta \Big|_{\pi/3}^{0} \right) \\
&= -\sqrt{3} \\
&\approx -1.73.
\end{aligned}$$