Due Date: April 20, 2012 Friday class time NAME:..

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Math 114 Calculus – Homework 4 – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are 4 questions on your booklet. Write your name on top of every page.
Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.
Everything you write on your paper should be part of a well constructed sentence. No hanging equations will be read. No sequence of equations will be read unless they are part of a well constructed, meaningful sentence.

Q-1) Read Theorem 8 on page 731. Then examine Example 3 on page 738. Now show that the equation $\dfrac{1+x+y^3}{1+x^3+y^4}=1$ has a solution of the form $y=f(x)$ near $x=0$ satisfying $f(0)=1$, and find the terms up to fifth degree for the Taylor series for $f(x)$ in powers of x

Solution:

Set $F(x,y)=\dfrac{1+x+y^3}{1+x^3+y^4}-1$.

We check that $\dfrac{\partial F}{\partial y}(0,1)=-\dfrac{1}{2}\neq 0$, so y can be solved as $y=f(x)$ where f is an analytic function of x. Since $F(0,1)=0$, we must have $f(0)=1$. Thus the first few terms of the Taylor expansion of f around $x=0$ is

$$f(x)=1+ax+bx^2+cx^3+dx^4+ex^5+\cdots.$$

Note that $F(x,f(x))=0$ is equivalent to writing

$$(1+x+f(x)^3)-(1+x^3+f(x)^4)=0.$$

Expanding this in powers of x we obtain

$$(1-a)x-(b+3a^2)x^2-(c+6ab+3a^3+1)x^3-(3b^2+9a^2b+6ac+a^4)x^4 \\ -(6bc+9ab^2+e+9a^2c+6ad+4a^3b)x^5+\cdots\equiv 0.$$

Hence all coefficients are zero. Solving for a,b,c,d,e, we find that

$$y=1+x-3x^2+14x^3-85x^4+567x^5+\cdots.$$

NAME: STUDENT NO:

Q-2) Let $f(x, y, z) = (x^2 + y^2)\ln(1 + y^2) + yz + xz^3$. Let P_0 be the point $(1, 0, 2)$.

(i) Find the gradient of f at P_0.

(ii) Find the linearization of f at P_0.

(iii) Find the equation for the tangent plane at P_0 to the level surface of f through P_0.

(iv) If a bird flies through P_0 with speed 5, heading directly toward the point $(2, -1, 1)$, what is the rate of change of f as seen by the bird as it passes through P_0?

(v) In what direction fromP_0 should the bird fly at speed 5 to experience the greatest rate of increase of f?

Solution:

(i) $\nabla f = (2x\ln(1 + y^2) + z^3,\ 2y\ln(1 + y^2) + (x^2 + y^2)\,\frac{2y}{1+y^2} + z,\ y + 3xz^2)$.
$\nabla f(1, 0, 2) = (8, 2, 12)$.

(ii) $L(x, y, z) = f(1, 0, 2) + \nabla f(1, 0, 2) \cdot (x - 1, y - 0, z - 2) = 8x + 2y + 12z - 24$.

(iii) Equation of the tangent plane is $\nabla f(1, 0, 2) \cdot (x - 1, y - 0, z - 2) = 0$, which simplifies to $8x + 2y + 12z = 32$.

(iv) Let $\vec{u} = (2, -1, 1) - (1, 0, 2) = (1, -1, -1)$. Then $|\vec{u}| = \sqrt{3}$. The required rate of change is $5D_{\vec{u}}f(1, 0, 2) = 5\nabla f(1, 0, 2) \cdot (1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}) = 10\sqrt{3}$.

(v) The fastest rate of change is observed along the gradient $\nabla f(1, 0, 2) = (8, 2, 12)$.

NAME: STUDENT NO:

Q-3) Find all local/global minimum and maximum points of $f(x,y) = x^4 + 24y^2 - 4xy^3$, if they exist. Also find any saddle points if they exist.

Solution:

First find the critical points. $f_x = 0$ and $f_y = 0$ give $(0,0)$, $(2,2)$ and $(-2,-2)$ as critical points. The second derivative test immediately gives the points $(2,2)$ and $(-2,-2)$ as saddle points. The discriminant vanishes at $(0,0)$. Here we observe that

$$f(x,y) - f(0,0) = x^4 + 4y^2(6 - xy) > 0$$

for $|x|, |y| < 1$. So the origin is a local minimum point.

The function however is not bounded since $\lim_{x\to\infty} f(x,0) = \infty$ and $\lim_{y\to\infty} f(1,y) = -\infty$. So there is no global minimum or maximum.

NAME: STUDENT NO:

Q-4) Among all the ellipsoids of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which pass through the point $(2,1,3)$, find the ones with the minimum and the maximum volumes, if they exist.

Solution:

The volume of this ellipsoid is $V(a,b,c)=\dfrac{4}{3}\pi abc$, where we take $a,b,c>0$. The parameters a,b,c must satisfy the equation

$$f(a,b,c)=\frac{4}{a^2}+\frac{1}{b^2}+\frac{9}{c^2}-1=0.$$

Notice that for $b=2$ and any large a, we can find $c>1$ so that the equation $f(a,2,c)=0$ can be satisfied. But then $V(a,2,c)>(8\pi/3)a$ and is unbounded as a increases. Thus there is no maximum but we can look for minimum.

For this we use Lagrange multipliers method. From $\nabla V=\lambda\nabla f$ we obtain

$$\frac{a^3bc}{8}=\frac{ab^3c}{2}=\frac{abc^3}{18}=-\lambda,$$

from which we obtain

$$a=2b \quad\text{and}\quad c=3b.$$

Putting these into $f(a,b,c)=0$, we get $b=\sqrt{3}$. Hence the global minimum occurs at the point $(a,b,c)=(2\sqrt{3},\sqrt{3},3\sqrt{3})$ and then the ellipsoid with the maximum volume is

$$\frac{x^2}{12}+\frac{y^2}{3}+\frac{z^2}{27}=1$$

and has the volume

$$V(2\sqrt{3},\sqrt{3},3\sqrt{3})=24\sqrt{3}\pi\approx 131.$$