Date: 25 April 2012, Wednesday NAME:..
Time: 08:40-10:30
Ali Sinan Sertöz STUDENT NO:..

Math 114 Calculus II – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail, if that is necessary. A correct answer without proper reasoning may not get any credit, except in the fill-in-the-blanks type problems.

Q-1) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$.

Solution: We start with the geometric series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n, \quad \text{for } |x|<1.$$

By differentiating both sides term by term, we get

$$\left(\frac{1}{1-x}\right)'=\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}, \quad \text{for } |x|<1.$$

Now multiply both sides by x to obtain

$$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n, \quad \text{for } |x|<1.$$

Finally, putting $x=1/e$, we obtain

$$\sum_{n=1}^{\infty}\frac{n}{e^n}=\frac{1/e}{(1-1/e)^2}=\frac{e}{(e-1)^2}\approx 0.92.$$

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Q-2 The point $(0,0)$ is a critical point for the function $f(x,y) = x^4 + 24y^2 - 4xy^3 + 2012$. What is the nature of this critical point? Is it a local/global min/max or a saddle point?

Solution: The discriminant vanishes here so the second derivative test fails. But we observe that

$$f(x,y) - f(0,0) = x^4 + 4y^2(6 - xy) > 0$$

for $|x|, |y| < 1$. So the origin is a local minimum point.

The function however is not bounded since $\lim_{x\to\infty} f(x,0) = \infty$ and $\lim_{y\to\infty} f(1,y) = -\infty$. So there is no global minimum or maximum.

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Q-3) Define

$$f(x,y) = \begin{cases} \dfrac{y^5 - x^2 y}{x^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove or disprove that f is differentiable at the origin.

Solution:

We first calculate the first derivatives.

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0,$$
$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{x \to 0} \frac{y - 0}{y} = 1.$$

Next we check for differentiability. Let

$$\phi(x,y) = \frac{f(x,y) - f(0,0) - 0 \cdot x - 1 \cdot y}{\sqrt{x^2 + y^2}} = -2 \cdot \frac{x^2 y}{(x^2 + y^2)^{3/2}}.$$

Then

$$\lim_{x \to 0} \phi(x, \lambda x) = \frac{-2\lambda}{\sqrt{1 + \lambda^2}}.$$

Since this limit depends on path, the general limit does not exist and the function f is not differentiable at the origin.

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Q-4) Let $f(x,y)$, $\alpha(t)$ and $\beta(t)$ be C^∞ functions with the following data.

$$\begin{array}{cccccc} \alpha(0)=1, & \alpha'(0)=2, & \alpha''(0)=3, & \beta(0)=7, & \beta'(0)=11, & \beta''(0)=23, \\ f(1,7)=\pi, & f_x(1,7)=a, & f_{xx}(1,7)=u, & f_{xy}(1,7)=v, & f_{yy}(1,7)=w, & f_y(1,7)=b. \end{array}$$

Let $F(t)=f(\alpha(t),\beta(t))$.

(i) Find $F(0)$. *(1 point)*

(ii) Find $F'(0)$. *(4 points)*

(iii) Find $F''(0)$. *(15 points)*

Solution:

(i) $F(0)=f(\alpha(0),\beta(0))=f(1,7)=\pi$.

(ii) Set $x=\alpha(t)$ and $y=\beta(t)$. Then

$$\begin{aligned} F'(t) &= f_x(x,y)x' + f_y(x,y)y', \\ F'(0) &= a\cdot 2 + b\cdot 11 = 2a+11b. \end{aligned}$$

(iii) Taking the derivative of $F'(t)$ given above, we get

$$\begin{aligned} F''(t) &= [f_{xx}(x,y)x' + f_{xy}(x,y)y']x' + f_x(x,y)x'' + [f_{yx}(x,y)x' + f_{yy}(x,y)y']y' + f_y(x,y)y'', \\ F''(0) &= [u\cdot 2 + v\cdot 11]\cdot 2 + a\cdot 3 + [v\cdot 2 + w\cdot 11]\cdot 11 + b\cdot 23, \\ &= 3a + 23b + 4u + 44v + 121w. \end{aligned}$$

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Q-5) Change the order of integration of the following integrals as indicated.
(Grading: each box=1 point)

$$\int_0^8 \int_{x^2}^{6x+16} dy\, dx = \int_{\square}^{\square} \int_{\square}^{\square} dx\, dy + \int_{\square}^{\square} \int_{\square}^{\square} dx\, dy.$$

$$\int_{-4}^{0} \int_{-2x-5}^{\sqrt{25-x^2}} dy\, dx + \int_0^5 \int_{x-5}^{\sqrt{25-x^2}} dy\, dx = \int_{\square}^{\square} \int_{\square}^{\square} dx\, dy$$

$$+ \int_{\square}^{\square} \int_{\square}^{\square} dx\, dy$$

$$+ \int_{\square}^{\square} \int_{\square}^{\square} dx\, dy.$$

Solution:

$$\int_0^8 \int_{x^2}^{6x+16} dy\, dx = \int_{0}^{16} \int_{0}^{\sqrt{y}} dx\, dy + \int_{16}^{64} \int_{(y-16)/6}^{\sqrt{y}} dx\, dy.$$

$$\int_{-4}^{0} \int_{-2x-5}^{\sqrt{25-x^2}} dy\, dx + \int_0^5 \int_{x-5}^{\sqrt{25-x^2}} dy\, dx = \int_{-5}^{0} \int_{-(y+5)/2}^{y+5} dx\, dy$$

$$+ \int_{0}^{3} \int_{-(y+5)/2}^{\sqrt{25-y^2}} dx\, dy$$

$$+ \int_{3}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\, dy.$$