MATH 116 FINAL EXAM: INTERMEDIATE CALCULUS III, July 26, 2008

1. (20 points) Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) : 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution: Since $f(x, y)$ is a differentiable function in the whole xy-plane, the only places where $f(x, y)$ can assume absolute maximum and minimum values are
(i) points inside D where $f_x(x, y) = f_y(x, y) = 0$ (so called "critical points"') and
(ii) points on the boundary of D.

(i) Critical points:

$$\begin{aligned} f_x(x, y) &= 2x - 2y = 0 \\ f_y(x, y) &= -2x + 2 = 0 \end{aligned} \qquad \Longrightarrow \qquad (x, y) = (1, 1).$$

Thus, the only critical point is $(1, 1)$ and $f(1, 1) = 1$.

(ii) Boundary points: The boundary of D consists of four line segments: OA, AB, BC, CO, where $O = O(0, 0)$, $A = A(3, 0)$, $B = B(3, 2)$ and $C = C(0, 2)$.

(1) On the segment OA we have $y = 0$ and

$$f(x, y)|_{OA} = f(x, 0) = x^2, \quad 0 \le x \le 3,$$

is an increasing function whose values at the end points are $f(0, 0) = 0$ and $f(3, 0) = 9$.

(2) On the segment AB we have $x = 3$ and

$$f(x, y)|_{AB} = f(3, y) = 9 - 6y + 2y = 9 - 4y, \qquad 0 \le y \le 2,$$

is a decreasing function whose values at end points are $f(3, 0) = 9$ and $f(3, 2) = 1$.

(3) On the segment BC we have $y = 2$ and

$$f(x, y)|_{BC} = f(x, 2) = x^2 - 4x + 4 = (x - 2)^2, \qquad 0 \le x \le 3,$$

has a minimum value $f(2, 2) = 0$. Also values at the end points are $f(3, 2) = 1$, $f(0, 2) = 4$.

(4) On the segment OC we have $x = 0$ and

$$f(x, y)|_{OC} = f(0, y) = 2y, \qquad 0 \le y \le 2,$$

is an increasing function whose values at the end points are $f(0, 0) = 0$ and $f(0, 2) = 4$.

Conclusion:
The absolute maximum value of $f(x, y)$ on D is 9.
The absolute minimum value of $f(x, y)$ on D is 0.

2. (20 points) Let

$$\mathbf{F} = (3x^2y + z)\,\mathbf{i} + (x^3 + 2yz)\,\mathbf{j} + (x + y^2 + 4z^3)\,\mathbf{k}$$

be a vector field.
(a) Show that $\mathbf{F}$ is conservative.
(b) Find a potential function for $\mathbf{F}$.
(c) Evaluate the work integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is any smooth simple curve joining the points $A(0,1,1)$ to $B(1,1,0)$.

Solution:
(a) We have $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, where

$$M = 3x^2y + z, \qquad N = x^3 + 2yz, \qquad P = x + y^2 + 4z^3.$$

Since M, N, P have continuous first order partial derivatives and

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial N}{\partial z},$$

$$\frac{\partial M}{\partial z} = 1 = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y},$$

then, by the Component Test for Conservative Fields, $\mathbf{F}$ is conservative.
Remark: To prove that $\mathbf{F}$ is conservative one may also show that $\nabla\times\mathbf{F} = \vec{\mathbf{0}}$.

(b) Since $\mathbf{F}$ is conservative, then

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k},$$

where $f(x,y,z)$ is a potential function $\mathbf{F}$. We have,

1) $\frac{\partial f}{\partial x} = M = 3x^2y + z \Longrightarrow f(x,y,z) = x^3y + zx + g(y,z);$

2) $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y + zx + g(y,z)) = x^3 + \frac{\partial g(y,z)}{\partial y} = N = x^3 + 2yz \Longrightarrow \frac{\partial g(y,z)}{\partial y} = 2yz$
$\Longrightarrow g(y,z) = y^2z + h(z) \Longrightarrow f(x,y,z) = x^3y + zx + y^2z + h(z)$

3) $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^3y + zx + y^2z + h(z)) = x + y^2 + \frac{d\,h(z)}{d\,z} = P = x + y^2 + 4z^3 \Longrightarrow \frac{d\,h(z)}{d\,z} = 4z^3$
$\Longrightarrow h(z) = z^4 + Const \Longrightarrow f(x,y,z) = x^3y + zx + y^2z + z^4 + Const$

Thus, a potential function for $\mathbf{F}$ is $f(x,y,z) = x^3y + zx + y^2z + z^4 + Const.$

(c) Since $\mathbf{F}$ is conservative with a potential function $f(x,y,z) = x^3y + zx + y^2z + z^4 + Const$, then

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A) = f(1,1,0) - f(0,1,1) = -1.$$

3. (20 points) Verify the circulation-tangential form of Green's theorem for the field $\mathbf{F}(x,y) = xy\mathbf{i} + (y^2+x)\mathbf{j}$ over the unit circle $C : \vec{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$.

Solution: The Circulation-Curl form of Green's Theorem states that

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

for any simple closed piecewise smooth curve C, region R enclosed by C and for M and N being continuous together with their partial derivatives in some open region containing C and R.

<u>Evaluation of $\oint_C M dx + N dy$:</u>

With the parametrization $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$,

$$\oint_C M dx + N dy = \oint_C xy dx + (y^2+x) dy = \int_0^{2\pi} \{\cos t \sin t(-\sin t) + (\sin^2 t + \cos t)\cos t\} dt$$

$$= \int_0^{2\pi} \cos^2 t dt = \int_0^{2\pi} \frac{1+\cos(2t)}{2} dt = \pi.$$

<u>Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$:</u>

Region R is described in polar coordinates as $0 \le r \le 1$, $0 \le \theta \le 2\pi$. We have,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R \left(\frac{\partial}{\partial x}(y^2+x) - \frac{\partial}{\partial y}(xy) \right) dxdy = \iint_R (1-x)\, dxdy$$

$$= \int_0^{2\pi} \int_0^1 (1 - r\cos\theta) r dr d\theta = \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^3}{3}\cos\theta \right]_{r=0}^{r=1} = \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\theta \right) d\theta = \pi.$$

4. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where

$$\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$$

and $S : z = 4 - x^2 - y^2, z \geq 1$ and $\mathbf{n}$ points away from the origin.
a) (10 points) directly,
b) (10 points) by Stokes' theorem

Solution:

(a) We have,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R (\nabla \times \mathbf{F}) \cdot \frac{\nabla f}{|\nabla f \cdot \mathbf{p}|}\, dA,$$

where S is a level surface $f(x, y, z) = z + x^2 + y^2 - 4 = 0$ that lies above a plane region R in the xy-plane described by $x^2 + y^2 \leq 3$. Here, $\mathbf{p} = \mathbf{k}$. Also,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x - y)\mathbf{i} + (x - y)\mathbf{j},$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}, \qquad |\nabla f \cdot \mathbf{p}| = |1| = 1,$$

and

$$(\nabla \times \mathbf{F}) \cdot \frac{\nabla f}{|\nabla f \cdot \mathbf{p}|} = ((x - y)\mathbf{i} + (x - y)\mathbf{j}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) = 2(x^2 - y^2).$$

Thus,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R 2(x^2 - y^2) dx dy = \int_0^{2\pi} \int_0^{\sqrt{3}} 2(r^2 \cos^2\theta - r^2 \sin^2\theta) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} 2r^3 \cos(2\theta) dr d\theta = \int_0^{\sqrt{3}} \int_0^{2\pi} 2r^3 \cos(2\theta) d\theta dr = 0.$$

(b) By Stokes' Theorem,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C xz dx + yz dy + xy dz,$$

where $C: \quad \mathbf{r}(t) = \sqrt{3}\cos t\mathbf{i} + \sqrt{3}\sin t\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 2\pi$. We have,

$$\oint_C xz dx + yz dy + xy dz = \int_0^{2\pi} (\sqrt{3}\cos t(-\sqrt{3}\sin t) + \sqrt{3}\sin t(\sqrt{3}\cos t)) dt = 0.$$

5. (20 points) Find the surface integral of $f(x, y, z) = xy - z^2$ over the surface

$$S: \ \mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (u - v)\,\mathbf{j} + v\,\mathbf{k}, \quad (0 \le u \le 1,\ 0 \le v \le 1)$$

Solution: Since

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

and

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6},$$

then

$$\iint_S (xy - z^2) d\sigma = \int_0^1 \int_0^1 \{(u + v)(u - v) - v^2\} |\mathbf{r}_u \times \mathbf{r}_v| dudv = \sqrt{6} \int_0^1 \int_0^1 (u^2 - 2v^2) dudv = -\frac{\sqrt{6}}{3}.$$