Summer 2007-08 MATH 116 Homework 2

Due on July 2, 2008.
No late homework will be accepted.

1. Use the method of Lagrange multipliers to find the maximum and minimum values of the function

$$f(x, y) = x^2 + 2y^2 + 2x + 3$$

subject to the constraint $x^2 + y^2 = 4$.

Solution : Define $g(x, y) = x^2 + y^2$. First note that

$$\nabla f = \lambda \nabla g \iff (2x + 2)\mathbf{i} + (4y)\mathbf{j} = (2\lambda x)\mathbf{i} + (2\lambda y)\mathbf{j}$$

$$\iff 2x + 2 = 2\lambda x \text{ and } 4y = 2\lambda y$$

Hence $\nabla f = \lambda \nabla g$ implies $y = 0$ or $\lambda = 2$.

In case $y = 0$, the equality $g(x, y) = 4$ implies $x = \mp 2$.

In case $\lambda = 2$, the equality $2x + 2 = 2\lambda x$ implies $x = 1$ and therefore $g(x, y) = 4$ implies $y = \mp\sqrt{3}$.

Hence $\nabla f = \lambda \nabla g$ and $g(x, y) = 4$ are both satisfied only when $(x, y) = (2, 0)$, $(-2, 0)$, $(1, \sqrt{3})$, or $(1, -\sqrt{3})$.
We calculate the values of f at these points:
$f(2, 0) = 11$, $f(-2, 0) = 3$, $f(1, \sqrt{3}) = 12$, and $f(1, -\sqrt{3}) = 12$.

Since the set of points that satisfy $g(x, y) = 4$ is a closed and bounded set, we can say that the maximum value of $f(x, y)$ subject to $g(x, y) = 4$ is 12 and the minimum value of $f(x, y)$ subject to $g(x, y) = 4$ is 3.

2. Does the following limit exist

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1-x-y-xy}{x^2+y^2}.$$

Calculate the above limit if it exists. (Hint: You might want to use Taylor's formula for functions of two independent variables.)

Solution : Let $f(x,y)=e^{x+y}$. Then by Taylor's formula for $f(x,y)$ at the origin we have

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2}\left(x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right)$$

$$+\frac{1}{6}\left(x^3f_{xxx}(cx,cy)+3x^2yf_{xxy}(cx,cy)+3xy^2f_{xyy}(cx,cy)+y^3f_{yyy}(cx,cy)\right)$$

for some $0\le c=c(x,y)\le 1$. Note that

$$f_{xxx}(cx,cy)=f_{xxy}(cx,cy)=f_{xyy}(cx,cy)=f_{yyy}(cx,cy)=e^{c(x+y)}.$$

Hence

$$e^{x+y}=1+x+y+\frac{1}{2}\left(x^2+2xy+y^2\right)+\frac{1}{6}\left(x^3+3x^2y+3xy^2+y^3\right)e^{c(x,y)(x+y)}$$

for some $0\le c(x,y)\le 1$. Therefore

$$\frac{e^{x+y}-1-x-y-xy}{x^2+y^2}=\frac{1}{2}+\frac{1}{6}\left(\frac{x^3+3x^2y+3xy^2+y^3}{x^2+y^2}\right)e^{c(x,y)(x+y)}$$

for some $0\le c(x,y)\le 1$. Hence

$$\frac{1}{2}-\frac{1}{6}\left(|x|+3|y|+3|x|+|y|\right)e^{|x+y|}\le\frac{e^{x+y}-1-x-y-xy}{x^2+y^2}$$

and

$$\frac{e^{x+y}-1-x-y-xy}{x^2+y^2}\le\frac{1}{2}+\frac{1}{6}\left(|x|+3|y|+3|x|+|y|\right)e^{|x+y|}$$

we also note that

$$\lim_{(x,y)\to(0,0)}\frac{1}{2}\mp\frac{1}{6}\left(|x|+3|y|+3|x|+|y|\right)e^{|x+y|}=\frac{1}{2}.$$

Hence by the Sandwich Theorem

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1-x-y-xy}{x^2+y^2}=\frac{1}{2}.$$

3. Calculate $\int_2^3 \int_2^y \frac{\sin(x)}{x}\, dxdy + \int_3^4 \int_2^3 \frac{\sin(x)}{x}\, dxdy + \int_4^9 \int_{\sqrt{y}}^3 \frac{\sin(x)}{x}\, dxdy.$

Solution : Instead of the integral

$$\int_2^3 \int_2^y \frac{\sin(x)}{x}\, dxdy + \int_3^4 \int_2^3 \frac{\sin(x)}{x}\, dxdy + \int_4^9 \int_{\sqrt{y}}^3 \frac{\sin(x)}{x}\, dxdy$$

we can just write $\iint\limits_R \frac{\sin(x)}{x}\, dA$ where R is the region defined as follows:

$$R = \left\{ (x, y) \;\middle|\; \begin{array}{c} (2 \le y \le 3 \text{ and } 2 \le x \le y) \text{ or} \\ (3 \le y \le 4 \text{ and } 2 \le x \le 3) \text{ or} \\ (4 \le y \le 9 \text{ and } \sqrt{y} \le x \le 3) \end{array} \right\}$$

Hence

$$R = \left\{ (x, y) \mid 2 \le x \le 3 \text{ and } x \le y \le x^2 \right\}$$

Therefore instead of $\iint\limits_R \frac{\sin(x)}{x}\, dA$ we can now write

$$\int_2^3 \int_x^{x^2} \frac{\sin(x)}{x}\, dydx = \int_2^3 (x \sin x - \sin x)\, dx$$

$$= -x \cos x - \sin x + \cos x|_2^3 = -2\cos(3) - \sin(3) + \cos(2) + \sin(2).$$

4. Calculate the area of the region enclosed by the curve $r = \cos(2\theta)$.

Solution : The graph of $r = \cos(2\theta)$ is like a four leaved rose and the region enclosed by one of these loops is swept out by a ray that rotates from $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$. Hence the area enclosed by the curve $r = \cos(2\theta)$ is equal to

$$4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\cos(2\theta)} r \, dr \, d\theta = 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{cos^2(2\theta)}{2} \, d\theta = 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + cos(4\theta)}{4} \, d\theta =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + cos(4\theta)) \, d\theta = \theta + \frac{sin(4\theta)}{4} \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{2}$$

5. Calculate the improper integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dxdy.$$

Solution :

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r \, dr d\theta = \int_{0}^{2\pi} \lim_{b \to \infty} \int_{0}^{b} e^{-r^2} r \, dr \, d\theta =$$

$$= \int_{0}^{2\pi} \lim_{b \to \infty} \left(\frac{-e^{-r^2}}{2} \Big|_{0}^{b} \right) d\theta = \int_{0}^{2\pi} \lim_{b \to \infty} \left(\frac{-e^{-b^2}}{2} + \frac{1}{2} \right) d\theta = \int_{0}^{2\pi} \frac{1}{2} \, d\theta = \pi$$