1. Find the outward flux of the vector field
\[ \mathbf{F}(x, y) = (x^2 + y^2) \sin(x)i + e^{xy^2} \ln\left(\frac{y}{e}\right)j \]
across the rectangle with vertices (0, 1), (π, 1), (0, e), and (π, e).

**Solution:** Let \( C \) denote the rectangle with vertices (0, 1), (π, 1), (0, e), and (π, e).

We can consider \( C \) as the union of the following four line segments: \( C_1 \) from (0, 1) to (π, 1), \( C_2 \) from (π, 1) to (π, e), \( C_3 \) from (π, e) to (0, e), and \( C_4 \) from (0, e) to (0, 1).

Assume that \( \mathbf{n} \) denotes the outward-pointing unit normal vector on \( C \).

The curve \( C_1 \) can be parameterized by \( \mathbf{r}_1(t) = ti + j \) for \( t \) in \([0, \pi]\).
Hence on \( C_1 \) we have \( \mathbf{F} \cdot \mathbf{n} = \left( (t^2 + 1) \sin(t)i - e^tj \right) \cdot (-j) = e^t \).

The curve \( C_2 \) can be parameterized by \( \mathbf{r}_2(t) = \pi i + tj \) for \( t \) in \([1, e]\).
Hence on \( C_2 \) we have \( \mathbf{F} \cdot \mathbf{n} = \left( e^{\pi t^2} \ln\left(\frac{t}{e}\right)j \right) \cdot (i) = 0. \)

The curve \( C_3 \) can be parameterized by \( \mathbf{r}_3(t) = -t i + e j \) for \( t \) in \([-\pi, 0]\).
Hence on \( C_3 \) we have \( \mathbf{F} \cdot \mathbf{n} = \left( -(t^2 + e^2) \sin(t)i \right) \cdot (j) = 0. \)

The curve \( C_4 \) can be parameterized by \( \mathbf{r}_4(t) = -ti \) for \( t \) in \([-e, -1]\).
Hence on \( C_4 \) we have \( \mathbf{F} \cdot \mathbf{n} = \left( \ln\left(\frac{e^t}{e}\right)j \right) \cdot (-i) = 0. \)

Hence the outward flux of the vector field \( \mathbf{F}(x, y) \) across the rectangle \( C \) can be calculated as follows:

\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_3} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_4} \mathbf{F} \cdot \mathbf{n} \, ds
= \int_0^\pi e^t \, dt + 0 + 0 + e^\pi \bigg|_0^\pi = e^\pi - 1
\]
2. Find the work done by the force

\[ \mathbf{F}(x, y, z) = (2xz^3 + e^y)\mathbf{i} + (xe^y + 4y^3\cos(z))\mathbf{j} + (3x^2z^2 - y^4\sin(z))\mathbf{k} \]

over the curve parameterized by

\[ \mathbf{r}(t) = \sin(\pi t)\mathbf{i} + t^3\mathbf{j} + (2t - 1)\mathbf{k}, \quad \text{for } 0 \leq t \leq \frac{1}{2} \]

in the direction of increasing \( t \).

**Solution** : First we will check whether if the vector field \( \mathbf{F}(x, y, z) \) is conservative on \( \mathbb{R}^3 \). We have

\[
\begin{align*}
\frac{\partial}{\partial y} (2xz^3 + e^y) &= e^y = \frac{\partial}{\partial z} (xe^y + 4y^3 \cos(z)), \\
\frac{\partial}{\partial z} (xe^y + 4y^3 \cos(z)) &= -4y^3 \sin(z) = \frac{\partial}{\partial y} (3x^2z^2 - y^4 \sin(z)), \\
\frac{\partial}{\partial x} (2xz^3 + e^y) &= 6xz^2 = \frac{\partial}{\partial z} (3x^2z^2 - y^4 \sin(z)).
\end{align*}
\]

By the above three equalities we know that \( \mathbf{F}(x, y, z) \) is conservative on \( \mathbb{R}^3 \). Now let’s try to find a potential function \( f(x, y, z) \) for \( \mathbf{F}(x, y, z) \). We know that we should have \( f_x(x, y, z) = 2xz^3 + e^y \). Hence we have

\[ f(x, y, z) = x^2z^3 + xe^y + g(y, z) \]

for some function \( g(y, z) \). We also know that we should have \( f_y(x, y, z) = xe^y + 4y^3 \cos(z) \). This means \( g_y(y, z) = 4y^3 \cos(z) \). Hence we have

\[ f(x, y, z) = x^2z^3 + xe^y + y^4 \cos(z) + h(z) \]

for some function \( h(z) \). We also know that we should have \( f_z(x, y, z) = 3x^2z^2 - y^4 \sin(z) \). This means \( h'(z) = 0 \). Hence we have

\[ f(x, y, z) = x^2z^3 + xe^y + y^4 \cos(z) + C \]

for some constant \( C \). Hence the work done by the force \( \mathbf{F}(x, y, z) \) over the curve parameterized by \( \mathbf{r}(t) \) from \( t = 0 \) to \( t = \frac{1}{2} \) can be calculated as follows:

\[
\int_{\mathbf{r}(0)}^{\mathbf{r}(\frac{1}{2})} \mathbf{F} \, d\mathbf{r} = \int_{(0,0,-1)}^{(1,\frac{1}{8},0)} \mathbf{F} \, d\mathbf{r} = f(1, \frac{1}{8}, 0) - f(0, 0, -1) = \sqrt{e} + \frac{1}{4096}
\]
3. Among all smooth simple closed curves in the plane oriented counterclockwise, find the one along which the work done by 

\[ \mathbf{F}(x, y) = \left( \frac{4}{3}y^3 - 20y + 5 \right) \mathbf{i} + \left( 1 + 5x - 3x^3 \right) \mathbf{j} \]

is greatest and calculate the area of the region enclosed by this smooth simple closed curve.

**Solution :** Let \( C \) be a smooth simple closed curve in the plane oriented counterclockwise and \( R \) be the region enclosed by the curve \( C \). Let \( \mathbf{F}(x, y) = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} \). Then by Green’s Theorem we have

\[
\begin{align*}
(\text{Work done over } C) &= \oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_R \text{curl}(\mathbf{F}) \cdot \mathbf{k} \, dA = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \\
&= \iint_R \left( (5 - 9x^2) - (4y^2 - 20) \right) \, dA = \iint_R (25 - 9x^2 - 4y^2) \, dA = (\ast)
\end{align*}
\]

Define a region \( R_0 = \left\{ (x, y) \mid 9x^2 + 4y^2 \leq 25 \right\} \) and let \( C_0 \) be the boundary of the region \( R_0 \) oriented counterclockwise. Then we have

\[
(\ast) = \iint_{R \cap R_0} (25 - 9x^2 - 4y^2) \, dA + \iint_{R - R_0} (25 - 9x^2 - 4y^2) \, dA \\
\leq \iint_{R \cap R_0} (25 - 9x^2 - 4y^2) \, dA \leq \iint_{R_0} (25 - 9x^2 - 4y^2) \, dA =
\]

\[
= \oint_{C_0} \mathbf{F} \cdot \mathbf{T} \, ds = (\text{Work done over } C_0)
\]

Hence among all smooth curves the greatest work is done over \( C_0 \). Hence the area enclosed by \( C_0 \) is the area of \( R_0 \) which could be calculated by making the subs \( u = 3x \) and \( v = 2y \) as follows

\[
\iint_{R_0} dA = \iint_{\{(u, v) \mid u^2+v^2\leq 25\}} J(u, v) \, du \, dv = \iint_{\{(u, v) \mid u^2+v^2\leq 25\}} \frac{1}{6} \, du \, dv = \frac{25\pi}{6}
\]
4. Let $C$ be a smooth curve that encloses a region $R$ such that the area of the region $R$ is $7\pi$ and the interior of the region $R$ contains the rectangle $D = \{(x, y) \mid -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$.

Compute the outward flux of the vector field

$$
\mathbf{F}(x, y) = \left(\frac{2x + y}{x^2 + y^2} + 3x + 6y\right)\mathbf{i} + \left(\frac{2y - x}{x^2 + y^2} + 5x + 7y\right)\mathbf{j}
$$

across the curve $C$.

**Solution** : Define $D_0 = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and let $C_0$ be the boundary of $D_0$ oriented counterclockwise. Then $D_0$ is a region included in the region $D$ hence it is also included in the region $R$. First note that the vector field $\mathbf{F}(x, y)$ is defined on the region $R - D_0$ and second note the following equality

$$
\text{div}(\mathbf{F}) = \left(\frac{2x^2 + 2y^2 - 4x^2 - 2xy}{(x^2 + y^2)^2} + 3\right) + \left(\frac{2x^2 + 2y^2 - 4y^2 + 2xy}{(x^2 + y^2)^2} + 7\right) = 10
$$

Hence

$$
\iint_{R-D_0} \text{div}(\mathbf{F}) \, dA = 10 \iint_{R-D_0} dA = 10 \text{ (area of } R - D_0) =
$$

$$
= 10 \left(\text{ (area of } R) - \text{ (area of } D_0)\right) = 10(7\pi - \pi) = 60\pi
$$

We also have

$$
\oint_{C_0} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{C_0} -\left(\frac{2y - x}{x^2 + y^2} + 5x + 7y\right) \, dx + \left(\frac{2x + y}{x^2 + y^2} + 3x + 6y\right) \, dy =
$$

$$
= \oint_{C_0} -(2y - x + 5x + 7y) \, dx + (2x + y + 3x + 6y) \, dy = \oint_{C_0} -(4x + 9y) \, dx + (5x + 7y) \, dy =
$$

$$
= \iint_{D_0} (5 - (-9)) \, dA = 14 \iint_{D_0} dA = 14 \text{ (area of } D_0) = 14\pi
$$

Now we have

$$
\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds - \oint_{C_0} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{R-D_0} \text{div}(\mathbf{F}) \, dA
$$

thus the outward flux of the vector field $\mathbf{F}(x, y)$ across the curve $C$ is $74\pi$. 

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5. Find the area of the surface $z = 2xy$ inside the cylinder $x^2 + y^2 = 9$.

**Solution:** Define $f(x, y, z) = z - 2xy$ and let $S$ be the surface $f(x, y, z) = 0$ inside the cylinder $x^2 + y^2 = 9$. Choose $p = k$ and let $R$ be the shadow region of the surface $S$ in the $xy$-plane with unit normal vector $p$. Then

\[
\text{(Area of } S) = \iiint_S d\sigma = \iint_R \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \iint_R \frac{|-2yi - 2xj + k|}{|(-2yi - 2xj + k) \cdot k|} dA =
\]

\[
= \iint_R \sqrt{4y^2 + 4x^2 + 1} dA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1} r dr d\theta =
\]

\[
= \int_0^{2\pi} \left( \frac{1}{12} (4r^2 + 1)^{\frac{3}{2}} \right) \bigg|_0^3 d\theta = \int_0^{2\pi} \frac{1}{12} \left( 37^{\frac{3}{2}} - 1 \right) d\theta \approx \frac{\pi}{6} \left( 37^{\frac{3}{2}} - 1 \right)
\]