Solution to MATH 116 Midterm II Exam, July 5, 2008

1. (15 points) Calculate

$$\iint_R (3x+1)\,dA$$

where R is the region bounded by $y=x^2$, $y=(x-1)^2$, $y=0$.
Solution:

The region of integration is a "triangular" region bounded by

1) the part of the parabola $y=x^2$ from $(0,0)$ to $(1/2,1/4)$ on the left;

2) the line segment connecting $(0,0)$ and $(1,0)$ from below;

3) the part of the parabola $y=(x-1)^2$ from $(1/2,1/4)$ to $(1,0)$ on the right.

1^{st} way:

$$\iint_R (3x+1)\,dA = \int_0^{1/4}\int_{\sqrt{y}}^{1-\sqrt{y}} (3x+1)dx\,dy = \int_0^{1/4} \left(\frac{3x^2}{2}+x\right)\Bigg|_{x=\sqrt{y}}^{x=1-\sqrt{y}} dy$$

$$= \int_0^{1/4}\left(\frac{5}{2}-5\sqrt{y}\right)dy = \left(\frac{5}{2}y - 5\frac{y^{3/2}}{3/2}\right)\Bigg|_{y=0}^{y=1/4} = \frac{5}{24}.$$

2^{nd} way:

$$\iint_R (3x+1)\,dA = \int_0^{1/2}\int_0^{x^2} (3x+1)dy\,dx + \int_{1/2}^1\int_0^{(x-1)^2} (3x+1)dy\,dx =$$

$$= \int_0^{1/2} (3x+1)y\Bigg|_{y=0}^{y=x^2} dx + \int_{1/2}^1 (3x+1)y\Bigg|_{y=0}^{y=(x-1)^2} dx$$

$$= \int_0^{1/2}(3x^3+x^2)dx + \int_{1/2}^1 (3x^3-5x^2+x+1)dx$$

$$= \left(\frac{3}{4}x^4+\frac{x^3}{3}\right)\Bigg|_0^{1/2} + \left(\frac{3}{4}x^4 - \frac{5}{3}x^3 + \frac{x^2}{2}+x\right)\Bigg|_{1/2}^1 = \frac{5}{24}.$$

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2. a) (15 points) Find the area of the region between the cardioid $r = 1 + \cos\theta$ and the circle $r = \cos\theta$.

Solution:

The area of the region R_1 inside the cardioid is

$$Area(R_1) = \int_0^{2\pi} \int_0^{1+\cos\theta} r dr\, d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=0}^{r=1+\cos\theta} d\theta = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{4} + \cos\theta + \frac{\cos(2\theta)}{4}\right) d\theta = \frac{3\pi}{2} \quad square\, units.$$

The area of the disc R_2 enclosed by the circle $r = \cos\theta$ (whose equation is $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ in the Cartesian coordinates) is

$$Area(R_2) = \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \quad square\, units.$$

Thus, the area of the region inside the cardioid and outside of the circle is

$$Area(R_1) - Area(R_2) = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4} \quad square\, units.$$

2. b) (10 points) Calculate the improper integral

$$\int_0^{\infty} \int_x^{\sqrt{3}x} e^{-(x^2+y^2)}\, dy\, dx.$$

Solution:

The region of integration is the angle in the first quadrant between the lines $y = x$ and $y = \sqrt{3}x$. These two boundary lines in polar coordinates have equations $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$.

In polar coordinates the integral can be rewritten and calculated as

$$\int_{\pi/4}^{\pi/3} \int_0^{\infty} e^{-r^2} r dr d\theta = \frac{1}{2} \int_{\pi/4}^{\pi/3} \{\lim_{c\to\infty} \int_0^c e^{-r^2} r dr\, \} d\theta = \int_{\pi/4}^{\pi/3} \lim_{c\to\infty} \left(-\frac{e^{-r^2}}{2}\right)\Big|_0^c d\theta =$$

$$= \int_{\pi/4}^{\pi/3} \lim_{c\to\infty} \left(-\frac{e^{-c^2}}{2} + \frac{1}{2}\right) d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2} d\theta = \frac{\pi}{24}.$$

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3. a) (10 points) Use Taylor's formula for $f(x,y) = \int_0^{x+y^2} e^{-t^2}dt$ at the origin to find a quadratic approximation of $f(x,y)$ near the origin.

Solution: The quadratic approximation of $f(x,y)$ at the origin is $f(x,y) \approx Q(x,y)$, where

$$Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2!}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2).$$

Since $f(0,0) = 0$,

$$f_x = e^{-(x+y^2)^2}, \qquad f_x(0,0) = 1,$$

$$f_y = e^{-(x+y^2)^2} \cdot 2y, \qquad f_y(0,0) = 0,$$

$$f_{xx} = -e^{-(x+y^2)^2} \cdot 2(x+y^2), \qquad f_{xx}(0,0) = 0,$$

$$f_{xy} = -e^{-(x+y^2)^2} \cdot 2(x+y^2) \cdot 2y, \qquad f_{xy}(0,0) = 0,$$

$$f_{yy} = 2e^{-(x+y^2)^2} - 2ye^{-(x+y^2)^2} \cdot 2(x+y^2) \cdot 2y, \quad f_{yy}(0,0) = 2,$$

then

$$Q(x,y) = x + y^2.$$

3. b) (10 points) Use the method of Lagrange multipliers to find the volume of the largest (maximum volume) closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1$.

Solution: We have to maximize the function $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = \frac{x}{3} + \frac{y}{4} + \frac{z}{2} - 1 = 0$. To do so, we have to solve the system

$$\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{array} \quad \rightarrow \quad \begin{array}{l} yz\vec{i} + xz\vec{j} + xy\vec{k} = \frac{\lambda}{3}\vec{i} + \frac{\lambda}{4}\vec{j} + \frac{\lambda}{2}\vec{k} \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1 \end{array} \quad \rightarrow \quad \begin{array}{l} 3yz = \lambda \\ 4xz = \lambda \\ 2xy = \lambda \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1 \end{array}$$

$$\begin{array}{l} z(3y-4x) = 0 \\ 2x(2z-y) = 0 \\ \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1 \end{array} \quad \rightarrow (since \quad x,y,z \neq 0) \quad \begin{array}{l} x = (3/4)y \\ z = (1/2)y \\ 1 = \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = \frac{1}{3} \cdot \frac{3}{4}y + \frac{1}{4}y + \frac{1}{4}y = \frac{3}{4}y \end{array}$$

Thus, the dimensions that maximize the volume of the rectangle are

$$y = \frac{4}{3}, \qquad x = \frac{3y}{4} = 1, \qquad z = \frac{y}{2} = \frac{2}{3},$$

and therefore, the volume of the largest rectangular box satisfying conditions above is

$$Maximum \quad Volume \quad = 1 \cdot \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9} \quad cubic\,units.$$

Remark: We consider a continuous function $f(x,y,z) = xyz$ on a closed bounded triangular region R described by $\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$. Therefore, $f(x,y,z)$ attains its absolute maximum and minimum values on R. At all the boundary points of R function $f(x,y,z)$ takes its absolute minimum value 0. In the solution above we have found that the absolute maximum value $\frac{8}{9}$ of $f(x,y,z)$ is attained at the interior point $\left(1, \frac{4}{3}, \frac{2}{3}\right)$.

4. Let D be the solid bounded above by the sphere $x^2+y^2+z^2=4$ and below by the paraboloid $z=\frac{1}{3}(x^2+y^2)$. Without evaluating the integrals, set up iterated integrals in the following coordinate systems to calculate the volume of D:

Solution:
The boundary surfaces are

Coordinates :	*Cartesian*	*Cylindrical*	*Spherical*
from above :	$x^2+y^2+z^2=4$	$r^2+z^2=4$	$\rho=2$
from below :	$3z=x^2+y^2$	$3z=r^2$	$3\cos\phi=\rho\sin^2\phi$

The boundary surfaces meet at the points (x,y,z), where $x^2+y^2+z^2=4$ and $3z=x^2+y^2$. Therefore, z coordinate of the points on the intersection curve satisfies $3z+z^2=4$, $z>0$, or the same, $z=1$. It implies that x and y coordinates of the points on the intersection curve satisfy $x^2+y^2=3$.

Thus, <u>the orthogonal xy-projection</u> of the solid D is a disc enclosed by the circle $x^2+y^2=3$.

Also note that all the points (ρ,ϕ,θ) on the intersection curve satisfy $\rho=2$ and $\phi=\frac{\pi}{3}$.

(a) (5 points) in cartesian coordinates,

$$Volume(D)=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

(b) (5 points) in cylindrical coordinates, and

$$Volume(D)=\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\frac{1}{3}r^2}^{\sqrt{4-r^2}} r \quad dz\,dr\,d\theta$$

(c) (10 points) in spherical coordinates.
We divide the solid D into two parts: D_1 and D_2, where D_1 is a solid bounded below by the cone $\phi=\frac{\pi}{3}$ and above by the sphere $\rho=2$, and D_2 is a solid bounded below by the paraboloid $3\cos\phi=\rho\sin^2\phi$ and above by the cone $\phi=\frac{\pi}{3}$.

$$Volume(D)=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2}\rho^2\sin\phi \quad d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{3\frac{\cos\phi}{\sin^2\phi}}\rho^2\sin\phi \quad d\rho\,d\phi\,d\theta$$

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5. (20 points) Evaluate the integral

$$\iint_R \cos\left(\frac{2x+2y}{x-y}\right)\, dA$$

where R is the trapezoidal (yamuksu) region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.
Solution: Introduce

$$u = x + y,$$

$$v = x - y.$$

Then

$$x = \tfrac{1}{2}(u+v),$$

$$y = \tfrac{1}{2}(u-v),$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Boundary curves in xy − plane	*Boundary curves in uv − plane*
$x = 0$	$u = -v$
$y = 0$	$u = v$
$x - y = 1$	$v = 1$
$x - y = 2$	$v = 2$

Thus the region of integration in uv-plane is a trapezoidal region bounded by
1) $u = -v$ on the left;
2) $u = v$ on the right;
3) $v = 1$ from below;
1) $v = 2$ from above.
Then

$$\iint_R \cos\left(\frac{2x+2y}{x-y}\right) dA = \int_1^2 \int_{-v}^{v} \cos\left(\frac{2u}{v}\right) |J(u,v)| du dv = \frac{1}{2}\int_1^2 \int_{-v}^{v} \cos\left(\frac{2u}{v}\right) du dv$$

$$= \frac{1}{2}\int_1^2 \frac{v}{2} \sin\left(\frac{2u}{v}\right)\Big|_{u=-v}^{u=v} dv = \frac{\sin 2}{2}\int_1^2 v dv = \frac{3}{4}\sin 2.$$