MATH 116 REVIEW PROBLEMS for the FINAL EXAM

The following questions are taken from old final exams of various calculus courses taught in Bilkent University

1. Consider the line integral $\oint_C (2xy^2z + y)dx + 2x^2yzdy + (x^2y^2 - 2z)dz$ where $C : x = \cos t,\ \ y = \sin t,\ \ z = \sin t$ from $t = 0$ to $t = 2\pi$.
a) Evaluate the above line integral directly.
b) Evaluate the above line integral by using Stokes' theorem.

2. Evaluate the line integral $\oint_C (-yx^2 + \sin x^2)dx + (xy^2 + e^{y^2})dy$ where C is the boundary of the region in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $y = x$, $y = \sqrt{3}x$ traced in the counterclockwise sense.

3. Consider the solid in $\mathbb{R}^3$ which is bounded by $z = 0$, $z = 1 + x^2 + y^2$, $x^2 + y^2 = 4$. Express the volume of this solid as an iterated triple integral (or sum and/or difference of iterated triple integrals) in the following coordinate systems. But do not evaluate.
a) In xyz-coordinates.
b) In cylindrical coordinates.
c) In spherical coordinates.

4. a) Let f be a function of class C^1 in $\mathbb{R}^2$ and $z = f(x, y)$ where $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$ for some constant angle α. Show that

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

b) Let f be of class C^2 in $\mathbb{R}^2$ and $z = f(u, v)$ where $u = x^2 + yx$, $v = x - 2y^2$. Given that

$$\left.\frac{\partial f}{\partial u}\right|_{u=6,v=0} = 3,\quad \left.\frac{\partial f}{\partial v}\right|_{u=6,v=0} = -5,\quad \left.\frac{\partial^2 f}{\partial u^2}\right|_{u=6,v=0} = -3,\quad \left.\frac{\partial^2 f}{\partial u\partial v}\right|_{u=6,v=0} = 2,\quad \left.\frac{\partial^2 f}{\partial v^2}\right|_{u=6,v=0} = 1,$$

find $\left.\dfrac{\partial^2 z}{\partial x^2}\right|_{x=2,y=1}$.

5. Evaluate

$$\int_0^1 \int_{\sqrt{y}}^1 x^3 e^{yx}\,dx\,dy.$$

6. Consider

$$I(C) = \oint_C \frac{x - y}{x^2 + y^2}dx + \frac{x + y}{x^2 + y^2}dy$$

where C is an arbitrary piecewise smooth simple closed curve which does not pass through the origin and traced counterclockwise. Calculate $I(C)$
a) when origin is not enclosed by C,
b) when origin is enclosed by C.

7. Consider the region D in the upper half space of $\mathbb{R}^3$ (i.e. $z \geq 0$) bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the upper nappe of the cone $z^2 = b^2(x^2 + y^2)$ where $a > 0$, $b > 0$ are constants. Write the volume of D as a triple integral (or sum of triple integrals) in
a) x, y, z coordinates,
b) cylindrical coordinates,
c) spherical coordinates.
Do not evaluate the integrals.

8. Consider the surface integral

$$I = \iint_S curl\ \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$$

where S is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, $\vec{\mathbf{n}}$ is the unit normal vector to S that points away from the origin and $\vec{\mathbf{F}}(x, y, z) = y\,\vec{\mathbf{i}} + z\,\vec{\mathbf{j}} + x\,\vec{\mathbf{k}}$.
a) Evaluate I directly.
b) Evaluate I by using Stokes' Theorem or Divergence Theorem.

9. Consider the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ where $a > 0, b > 0, c > 0$ are constants. Assume this plane passes through the point $(2, 1, 2)$ and cuts off the smallest volume from the first octant. Find a, b, c.

10. Find the volume of the solid whose base is the region in the xy-plane between the circles $x^2 + y^2 = 2y$ and $x^2 + y^2 = 3y$ and whose top lies in the plane $z = 3 - y$.

11. Evaluate

$$\oint_C \left(\frac{y^3}{x^2} + \sin x^2\right) dx + (y^3 \ln x + e^{y^2})dy$$

where C is the boundary of the region in the first quadrant bounded by the hyperbolas $xy = 1, xy = 3$ and the lines $y = x, y = 2x$.

12. Let S denote the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$. If $\vec{\mathbf{a}} = a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}$ is a constant vector and $\vec{\mathbf{F}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}} + z\vec{\mathbf{k}}$, find

$$\iint_S \text{curl}(\vec{\mathbf{a}} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS,$$

where $\vec{\mathbf{n}}$ is the outer unit normal to S.

13. The plane $2x + 4y + z = 15$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the highest and the lowest points on this ellipse (i.e., the points with the largest and smallest z-coordinate).

14. Evaluate the following line integral:

$$\oint_C (-x^2y + e^{x^2})dx + (xy^2 + \sin y^2)dy$$

where C is the boundary of the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $y = \frac{2}{x}$ and $y = \frac{4}{x}$ traced once in the positive (i.e. counterclockwise) direction.

15. Find the volume of the solid in the upper space(i.e. $z \geq 0$) bounded by the plane $z = 0$, the cylinder $x^2 + (y-1)^2 = 1$ and the cone $z^2 = x^2 + y^2$.

16. a) Evaluate the surface integral $\iint_S z dS$ where S is the part of the cylinder $y^2 + z^2 = 9$ in the first octant bounded by $x = 0$ and $x = 4$.
b) Let $\vec{\mathbf{F}} = M(x,y)\vec{\mathbf{i}} + N(x,y)\vec{\mathbf{j}}$ be a vector field in the plane such that M and N are of class C^1 and $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \geq 1$ for all (x,y) in the plane. Show that there is no simple closed curve in the plane whose tangent vector is parallel to $\vec{\mathbf{F}}$ at all of its points.

17. Let $\pi : ax + by + cz = k$ with $a^2 + b^2 + c^2 = 1$ be a plane in the 3-space and let C be a simple closed curve lying in the plane π. Show that

$$\left| \frac{1}{2} \oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz \right|$$

is equal to the area of the region on π enclosed by C.

18. Assume $f : R^2 \to R$ is such that $f(x,y)$ depends only on the distance r of (x,y) from the origin, i.e. $f(x,y) = g(r)$ where $r = \sqrt{x^2 + y^2}$.
a) Show that for all $(x,y) \neq (0,0)$ we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}g'(r) + g''(r).$$

b) Assume further that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

for all $(x,y) \neq (0,0)$. Use part a) and find $f(x,y)$.

19. a) Evaluate $\iint_D x \cos(x + y) dxdy$ where D is the triangular region with vertices at $(0,0), (\pi, 0), (\pi, \pi)$.

b) Show that

$$\int_0^c \int_0^y e^{m(c-x)} f(x) dx dy = \int_0^c (c-x) e^{m(c-x)} f(x) dx$$

where c and m are constants and $c > 0$.

20. Let D be the solid in R^3 bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the cone $z = \sqrt{x^2 + y^2}$. Write the volume of D as an iterated triple integral in
a) Cartesian coordinates (i.e. x, y, z coordinates),
b) Cylindrical coordinates,
c) Spherical coordinates,
d) Use a) or b) or c) above and compute the volume of D.

21. Assume $h : R^2 \to R$ and $k : R^2 \to R$ have continuous first order partial derivatives in R^2. Assume also at every point (x, y) of the circle $C : x^2+y^2 = 1$, we have $h(x, y) = 1, k(x, y) = y$. Define two vector fields $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ in R^2 as follows:

$$\vec{\mathbf{F}}(x, y) = k(x, y)\vec{\mathbf{i}} + h(x, y)\vec{\mathbf{j}}, \quad \vec{\mathbf{G}}(x, y) = \left(\frac{\partial h}{\partial x} - \frac{\partial h}{\partial y}\right)\vec{\mathbf{i}} + \left(\frac{\partial k}{\partial x} - \frac{\partial k}{\partial y}\right)\vec{\mathbf{j}}.$$

Find the value of the double integral $\iint_D \vec{\mathbf{F}} \cdot \vec{\mathbf{G}} \, dxdy$ where D is the disc $x^2 + y^2 \leq 1$.

21. a) Compute the surface area of the paraboloid $x^2 + z^2 = 3ay$ which is cut off by the plane $y = a$.
b) Let S be the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and

$$\vec{\mathbf{F}} = (x + yz^2)\vec{\mathbf{i}} + (2y + x^3z)\vec{\mathbf{j}} + (x^2 + y^2)\vec{\mathbf{k}}.$$

Find the flux of $\vec{\mathbf{F}}$ across S in the direction of the normal which points away from the origin.

22. Find the absolute minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints

$$y + 2z = 12 \text{ and } x + y = 6.$$

23. a) Evaluate the following double integral

$$I = \int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \cos(y^2) \, dy \, dx.$$

b) Let D be the solid in the first octant bounded below by the xy-plane, above by the cone $z = \sqrt{x^2 + y^2}$, on the sides by yz-plane and the cylinder $x^2 + (y - 1)^2 = 1$. Find the volume of D.

24. Let C be a smooth simple closed curve lying in the set $S = \{(x, y) : 1 < x^2 + y^2 < 9\}$ which is traversed once in the counterclockwise direction. Find all possible values of

$$I(C) = \oint_C \frac{y}{x^2 + y^2} \, dx - \frac{x}{x^2 + y^2} \, dy.$$

25. Let $\pi : Ax + By + Cz = k$ be a plane in the space such that $A \neq 0$, $B \neq 0$ and $C \neq 0$. Let S be a bounded set lying on π which has an area, and let S_1, S_2, S_3 denote its projections on the three coordinate planes. Show that

$$a(S) = \sqrt{a(S_1)^2 + a(S_2)^2 + a(S_3)^2}.$$

26. Let $S:\ z = 9 - x^2 - y^2,\ z \geq 0$, i.e. S is the part of the paraboloid in the upper space. Let

$$\vec{\mathbf{F}} = yz^4\,\vec{\mathbf{i}} + xz^3\,\vec{\mathbf{j}} + (x^2 + y^2)\,\vec{\mathbf{k}}.$$

Find the flux of $\vec{\mathbf{F}}$ across S in the direction that points away from the origin.

27. Evaluate the following line integral

$$\int_C \frac{x^2}{1+y}\,dx + e^{xy}\,x\,dy$$

where C is the curve $y = x^2$ from the point $A(0,0)$ to the point $B(1,1)$.

27. Evaluate the line integral

$$\int_C 2\,\cos y\,dx + \left(\frac{1}{y} - 2x\,\sin y\right)\,dy + \frac{1}{z}\,dz$$

where C is the curve of intersection of the surfaces $(8-\pi)x + 2y - 4z = 0$ and $16z = (32 - \pi^2)x^2 + 4y^2$ from the point $A(0,2,1)$ to the point $B(1,\pi/2,2)$.

28. By using the Stokes' theorem, evaluate

$$\int_C (y - z)\,dx + (z - x)\,dy + (x - y)\,dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $\frac{x}{3} + \frac{z}{4} = 1$ traversed in the counterclockwise sense when viewed from high above the xy-plane.