Date: July 25, 2009, Saturday
Time: 10:00-12:00

NAME:..

STUDENT NO:..

SECTION NUMBER: ....

Math 116 Intermediate Calculus III – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit. Without the correct **section number**, your grade may not be entered in SAPS.

Q-1-a) Find the linear approximation of the function $f(x,y) = 3x^2 + 2xy - 4y^2$ at $(1,1)$.
Q-1-b) Estimate an upper bound for the absolute value of the error made by this linear approximation in $D = \{(x,y) \in \mathbb{R}^2 \mid |x-1| \leq 0.1, |y-1| \leq 0.1\}$.

Solution 1-a:

$$f_x = 6x + 27, \quad f_y = 2x - 8y$$

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = 8x - 6y - 1$$

Hence the linear approximation of f is

$$f(x,y) \approx 8x - 6y - 1.$$

Solution 1-b: $|E| \leq \frac{M}{2}(|x-1| + |y-1|)^2$ leads to $|E| \leq 0.16$. Here $M = 8$ since

$$f_{xx} = 6, \quad f_{yy} = -8, \quad f_{xy} = 2$$

and M is an upper bound for these second partial derivatives in D.

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Q-2-a) Evaluate $\int_0^4 \int_{x/2}^{\sqrt{x}} e^{3y^2-y^3}\, dy\, dx$. *(8 points)*

Solution

$$\begin{aligned}
\int_0^4 \int_{x/2}^{\sqrt{x}} e^{3y^2-y^3}\, dy\, dx &= \int_0^2 \int_{y^2}^{2y} e^{3y^2-y^3}\, dx\, dy \\
&= \int_0^2 (2y-y^2) e^{3y^2-y^3}\, dz \\
&= \frac{1}{3}\left(e^{3y^2-y^3}\Big|_0^2 \right) \\
&= \frac{1}{3}(e^4-1).
\end{aligned}$$

Q-2-b) Let V denote the volume of the solid region which lies inside the sphere $x^2+y^2+z^2=4$ and outside the cylinder $x^2+y^2=1$. Fill in the following boxes. (Do not evaluate any integral!) *(12 points)*

(i) V in cylindrical coordinates is $= 2 \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} r\, dz\, dr\, d\theta.$

(ii) V in spherical coordinates is $= 2 \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

Solution:

(i) V in cylindrical coordinates is $= 2 \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$

(ii) V in spherical coordinates is $= 2 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_{1/\sin\phi}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

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Q-3) Let $\mathbf{F} = [\frac{35x+17y}{x^2+y^2} + y]\mathbf{i} + [\frac{-17x+35y}{x^2+y^2} + 3x]\mathbf{j}$ be a vector field over a planar region D bounded by a simple curve C. Let the area of D be 28π and assume that D contains a disc of radius 5 centered at the origin. Find the circulation of the vector field $\mathbf{F}$ around C.

Solution: Let $\mathbf{F} = [M, N]$. Then $N_x - M_y = 2$.

Let S be the circle of radius 5, centered at the origin and oriented counterclockwise. Let R be the region between C and S. Clearly the area of R is 3π and its boundary is $C - S$.

By Green's Theorem we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{T}\,ds - \oint_S \mathbf{F}\cdot\mathbf{T}\,ds &= \oint_{C-S} \mathbf{F}\cdot\mathbf{T}\,ds \\ &= \int\int_R (N_x - M_y)\,dxdy \\ &= \int\int_R 2\,dxdy \\ &= 2\,\text{Area}(R) = 6\pi.\end{aligned}$$

This gives

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_S \mathbf{F}\cdot\mathbf{T}\,ds + 6\pi.$$

Using the usual parametrization $r(t) = (5\cos\theta, 5\sin\theta)$, $0 \le \theta \le 2\pi$ for S, we evaluate the circulation of $\mathbf{F}$ around S to be

$$\oint_S \mathbf{F}\cdot\mathbf{T}\,ds = \oint_S (M dx + N dy) = \int_0^{2\pi} (75\cos^2\theta - 25\sin^2\theta - 17)\,d\theta = 16\pi.$$

This gives

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 16\pi + 6\pi = 22\pi.$$

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Q-4) Evaluate the surface integral $\int\int_S x^2\, d\sigma$ where S is the part of the cone $x = \sqrt{y^2+z^2}$ between the planes $x = 0$ and $x = 1$.

Solution : Let R be the projection of the cone on yz-plane. Then $R = \{(y, z) \in \mathbb{R}^2 \mid y^2 + z^2 \leq 1\}$. Let $f(x, y, z) = \sqrt{y^2+z^2} - x$. The surface S is given by $f(x, y, z) = 0$ and

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{i}|}\, dA = \sqrt{1 + \frac{y^2}{y^2+z^2} + \frac{z^2}{y^2+z^2}}\, dA = \sqrt{2}\, dA.$$

Noting that $x^2 = y^2 + z^2$ on S, we have

$$\int\int_S x^2\, d\sigma = \int\int_R (y^2+z^2)(\sqrt{2})\, dA = \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta = \frac{\pi}{\sqrt{2}}.$$

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Q-5) Let $\mathbf{F} = \nabla \times \mathbf{E} + z^3\,\mathbf{k}$ be a vector field with $E = e^{yx^2}\,\mathbf{i} + e^{zy^2}\,\mathbf{j} + e^{xz^2}\,\mathbf{k}$. Find the outward flux of $\mathbf{F}$ through the unit sphere whose center is at the origin.

Solution: Let S denote this unit circle and D denote the unit ball that S contains. Let $\vec{n}$ be the unit normal vector field of S pointing outwards. Then use divergence theorem:

$$\begin{aligned}
\text{Flux} = \int\!\!\int_S \mathbf{F}\cdot\vec{n}\,d\sigma &= \int\!\!\int\!\!\int_D \nabla\cdot\mathbf{F}\,dV \\
&= \int\!\!\int\!\!\int_D (3z^2)\,dV \quad (\text{since } \nabla\cdot\nabla\times E = 0 \text{ for any } E) \\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\right)\left(\int_0^1 \rho^4\,d\rho\right) \\
&= 3(2\pi)\left(\frac{2}{3}\right)\left(\frac{1}{5}\right) \\
&= \frac{4\pi}{5}.
\end{aligned}$$