Due date: July 10, 2009, Friday

Math 116 Calculus – Homework # 3 – Solutions

Q-1) Consider the space curve C given by the parametrization:
$x(t) = \frac{t^2}{2}$, $y(t) = \frac{2\sqrt{2}t^{3/2}}{3}$, $z(t) = t$, $0 \le t \le 2$.
(i) Find the length of the curve C.
(ii) Find the arc length parametrization of C.

Solution: The length of the curve from 0 to t is given by the integral

$$s(t) = \int_0^t |v(\tau)|\,d\tau = \int_0^t (\tau + 1)\,d\tau = \frac{1}{2}t^2 + t.$$

Hence the length of the curve is $s(2) = 4$.

To find the arc length parametrization we must solve $s = \frac{1}{2}t^2 + t$ for t. This gives $t = -1 \pm \sqrt{1+2s}$. To decide on the sign we take into account that $s(0) = 0$ and $s(2) = 4$. This gives $t = -1 + \sqrt{1+2s}$. Putting this into the above parametrization of the curve gives the arc length parametrization:
$x(s) = \frac{(-1+\sqrt{1+2s})^2}{2}$, $y(s) = \frac{2\sqrt{2}(-1+\sqrt{1+2s})^{3/2}}{3}$, $z(s) = (-1+\sqrt{1+2s})$, $0 \le s \le 4$.

Q-2) Evaluate the line integral $\int_C f\,ds$ where C is given by the parametrization $x(t) = 1$, $y(t) = 2\cos t$, $z(t) = 2\sin t$, $0 \le t \le \pi$, and $f = f(x,y,z) = (x-1)(y^3\cosh z) + (y^2 + z^2 - 4)(x^3\sinh z) + y^2$.

Solution:

$$\int_C f\,ds = \int_0^\pi (f\,|_C)\,|v(t)|\,dt = 8\int_0^\pi \cos^2 t\,dt = 4\pi.$$

Q-3) Find the work done by the gradient of $x^2 + y^2 + z^2$ along the path $r(t) = (t, t^2, t^4)$, $0 \le t \le 1$.

Solution: Let $f = x^2 + y^2 + z^2$, and call the curve C. Then $\nabla f\,|_C = (2t, 2t^2, 2t^4)$, and $\nabla f \cdot dr = (2t + 4t^3 + 8t^7)\,dt$. Finally we have

$$\text{Work} = \int_C \nabla f \cdot dr = \int_0^1 (2t + 4t^3 + 8t^7)\,dt = 3.$$

Q-4) Find the flux of the position vector across the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution: The position vector is $F = (M, N) = (x, y)$, and the given ellipse C can be parameterized as $(x(t), y(t)) = (2\cos t, 3\sin t)$, $0 \le t \le 2\pi$. This gives

$$\text{Flux} = \int_C M dy - N dx = 6 \int_0^{2\pi} dt = 12\pi.$$

Q-5) Consider the vector field $F = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right)$.

(i) Show that F is conservative in the first octant.

(ii) Find a potential function for F.

(iii) Calculate the work done by F over the curve C which is parameterized as $x(t) = e^t$, $y(t) = t + \cos(\pi t/2)$, $z(t) = 1 + t^2$, $0 \le t \le 1$.

Solution:

(i) This is straightforward to check.

(ii) It is easy to show that $f(x, y, z) = \frac{1}{2} \ln(x^2 + y^2 + z^2) + k$ where k is any constant, gives all the potential functions for F.

(iii) Since F is conservative, its path integral depends only on the end points. Let $r(t)$ be the position vector of the path C. Then we have

$$\text{Work} = \int_C F \cdot dr = f(r(1)) - f(r(0)) = f(e, 1, 2) - f(1, 1, 1) = \frac{1}{2} \ln \left(\frac{e^2 + 5}{3} \right).$$