Due date: July 17, 2009, Friday

Math 116 Calculus – Homework # 4 – Solutions

Q-1) Let $\mathbf{F} = [\frac{ax+by}{x^2+y^2} + \alpha y]\mathbf{i} + [\frac{-bx+ay}{x^2+y^2} + \beta x]\mathbf{j}$ be a vector field over a region D bounded by a simple curve C. Here a, b, α and β are constants. Let the area of D be 50π and assume that D contains a disc of radius 7 centered at the origin. Find the value of b, in terms of a, α and β, so that the circulation of the vector field $\mathbf{F}$ around C is zero.

Solution: Let $\mathbf{F} = [M, N]$. Then $N_x - M_y = \beta - \alpha$.

Let S be the circle of radius 7, centered at the origin and oriented counterclockwise. Let R be the region between C and S. Clearly the area of R is π and its boundary is $C - S$.

By Green's Theorem we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{T}\,ds - \oint_S \mathbf{F}\cdot\mathbf{T}\,ds &= \oint_{C-S} \mathbf{F}\cdot\mathbf{T}\,ds \\
&= \int\int_R (N_x - M_y)\,dxdy \\
&= \int\int_R (\beta - \alpha)\,dxdy \\
&= (\beta-\alpha)\mathrm{Area}(R) = (\beta-\alpha)\pi.
\end{aligned}$$

This gives

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_S \mathbf{F}\cdot\mathbf{T}\,ds + (\beta-\alpha)\pi.$$

Using the usual parametrization $r(t) = (7\cos\theta, 7\sin\theta)$, $0 \le \theta \le 2\pi$ for S, we evaluate the circulation of $\mathbf{F}$ around S to be

$$\oint_S \mathbf{F}\cdot\mathbf{T}\,ds = \oint_S (M dx + N dy) = \int_0^{2\pi} (49\beta\cos^2\theta - 49\alpha\sin^2\theta - b)\,d\theta = 49\pi(\beta-\alpha) - 2\pi b.$$

This gives

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 50\pi(\beta-\alpha) - 2\pi b.$$

For this circulation to be zero, we must have

$$b = 25(\beta-\alpha).$$

Q-2) Evaluate $\int\int_S \mathbf{curl}(F) \cdot n \, d\sigma$, where S is the surface $z = 18 - 4x^2 - 9y^2$ with $z \geq 0$, oriented such that the unit normal vector points outward and F is the vector field

$$F = \left(z \cos x^2 + \frac{y}{9},\ z^2 \sin y + \frac{x}{3},\ 4x^2 + 9y^2 + z \sinh(x^2 + y^2) - 18\right).$$

Solution: We use Stokes' theorem

$$\int\int_S \mathbf{curl}(F) \cdot n \, d\sigma = \int_C F \cdot d\mathbf{r},$$

where C is the boundary of S parameterized by $\mathbf{r}(t) = (\frac{3}{\sqrt{2}} \cos\theta,\ \sqrt{2}\sin\theta,\ 0)$, $\theta \in [0, 2\pi]$.

The vector field F restricted to C takes the form $F = (\frac{1}{9}y,\ \frac{1}{3}x, 0) = (\frac{\sqrt{2}}{9}\sin\theta,\ \frac{1}{\sqrt{2}}\cos\theta, 0)$.

Then we have $F \cdot d\mathbf{r} = F \cdot (-\frac{3}{\sqrt{2}}\sin\theta,\ \sqrt{2}\cos\theta,\ 0)\, d\theta = (-\frac{1}{3}\sin^2\theta + \cos^2\theta)\, d\theta$. Thus we have

$$\int\int_S \mathbf{curl}(F) \cdot n \, d\sigma = \int_C F \cdot d\mathbf{r} = \int_0^{2\pi} (-\frac{1}{3}\sin^2\theta + \cos^2\theta)\, d\theta = \frac{2}{3}\pi.$$

Q-3) Find a closed curve C with counterclockwise orientation that maximizes the value of the integral

$$I = \oint_C \frac{y^3}{3} dx + (x - \frac{x^3}{3}) dy.$$

Solution: By Green's theorem we have

$$I = \int\int_R (1 - x^2 - y^2) dA,$$

where R is the region enclosed by C. The integral is maximal over $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$ which is the closed unit disk. Therefore C must be the unit circle with a counter clockwise parametrization $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$.

Q-4) Use Stokes' Theorem to evaluate the integral $\int_C \frac{y^2}{2}\, dx + zdy + xdz$, where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above.

Solution: Let S be the planar region contained inside C on the plane $x + z = 1$, and set $F = (M, N, P) = (y^2/2,\ z,\ x)$. Then

$$\int_C \frac{y^2}{2}\, dx + z\, dy + x\, dz = \int_C F \cdot dr$$

and Stokes' theorem says

$$\int_C F \cdot dr = \int\int_S \nabla \times F \cdot n\, d\sigma,$$

where $n = (1/\sqrt{2}, 0, 1\sqrt{2})$ is the unit normal vector of S pointing upwards to be compatible with the orientation of C.

$d\sigma$ is the area element on the surfaces S. Here we can take $f(x, y, z) = x + z - 1 = 0$ for S. Then $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot p|}\, dA$, where $p = (0, 0, 1)$ is the unit normal vector of the projection of S onto xy-plane. This gives $d\sigma = \sqrt{2}\, dA$.

We also have

$$\begin{aligned}
\nabla \times F \cdot n &= (P_y - N_z,\ M_z - P_x,\ N_x - M_y) \cdot n \\
&= (-1, -1, -y) \cdot (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \\
&= -\frac{1}{\sqrt{2}}(1 + y).
\end{aligned}$$

Let D be the projection of S onto xy-plane. We find its bounding curve by eliminating z from the equations $x + z = 1$ and $x^2 + 2y^2 + z^2 = 1$. This gives the circle $x^2 - x + y^2 = 0$.

Then we have

$$\begin{aligned}
\int\int_S \nabla \times F \cdot n\, d\sigma &= -\int\int_D (1 + y)\, dA \\
&= -\int\int_D dA - \int\int_D y\, dA \\
&= -\frac{\pi}{4},
\end{aligned}$$

Where the first integral gives the area of the circle while the second integral is zero since the odd function y is integrated around a symmetrical region around zero.

Q-5) Compute

$$\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where

$$\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$$

and $S : z = 4 - x^2 - y^2,\ z \geq 1$ and $\mathbf{n}$ points away from the origin.

a) directly, **b)** by Stokes' theorem

Solution-a: Let S be given by $f(x, y, z) = z + x^2 + y^2 - 4 = 0$. Let D be the projection of S onto xy-plane. Then D is the disk $x^2 + y^2 \leq 3$, and the unit normal of D is $p = (0, 0, 1)$. Then $\nabla \times \mathbf{F} = (x - y, x - y, 0)$, $\nabla f = (2x, 2y, 1)$, and

$$\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot \frac{\nabla f}{|\nabla f|} \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA = 2(x^2 - y^2)\, dA.$$

It follows that

$$\begin{aligned}
\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma &= 2 \int\!\!\int_D (x^2 - y^2)\, dA \\
&= 2 \int_0^{2\pi} \int_0^{\sqrt{3}} r^3 (\cos^2\theta - \sin^2\theta)\, dr\, d\theta \\
&= 2 \int_0^{2\pi} \cos 2\theta \int_0^{\sqrt{3}} r^3\, dr\, d\theta \\
&= 0.
\end{aligned}$$

Solution-b: By Stokes' theorem

$$\int\!\!\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the boundary of S and is parametrized as $\mathbf{r}(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 1)$, $0 \leq t \leq 2\pi$.

Here we have $\mathbf{F} \cdot d\mathbf{r} = 0$ on C, so the given integral is zero.

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