Date: July 27, 2009, Monday NAME:..................................................
Time: 14:00-16:00

STUDENT NO:..................................................

**SECTION NUMBER**: ..................................................

## Math 116 Intermediate Calculus III – Make-up Exam – Solutions

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

**PLEASE READ:**
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit. Without the correct **section number**, your grade may not be entered in SAPS.

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**Q-1)** Find the absolute minimum and absolute maximum values of the function

$$f(x,y) = 3x^2 + 12x + 4y^3 - 6y^2 + 5$$

on $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 + 4x \leq 0\}$.

**Solution :**

First we look for the critical points in the interior of D.

$f_x = 6x + 2 = 6(x+2)$ and $f_x = 0$ for $x = -2$.

$f_y = 12y^2 - 12y = 12y(y-1)$ and $f_y = 0$ when $y = 0$ or $y = 1$.

Therefore, the critical points of $f(x,y)$ inside $D$ is $(-2,0)$ and $(-2,1)$.

For the boundary of $D$, we use the relation $x^2 + 4x = -y^2$ and reduce $f(x,y)$ to $f(y) = 4y^3 - 9y^2 + 5$ , $-2 \leq y \leq 2$.

$f'(y) = 12y^2 - 18y = 6y(2y-3)$ and $f'(y) = 0$ when $y = 0$ or $y = 3/2$.

Together with the end points $y = \pm 2$, $f$ has 4 critical points on $[-2,2]$.

$f(0) = 5$, $f(3/2) = -7/4$, $f(2) = 1$, and $f(-2) = -63$. For the critical points in the interior, we find that $f(-2,0) = -7$ and $f(-2,1) = -9$.

Hence, the absolute maximum of $f$ is 5 and the absolute minimum of $f$ is -63.

NAME: STUDENT NO:

**Q-2)** Find the volume of the region that is both inside the paraboloid $z = x^2 + y^2$ and the cylindrical surface $x = 2y - y^2$, and from below bounded by the plane $z = 0$ and from above by the plane $z = 1$.

**Solution:**

This question is cancelled and the other questions are graded over 25 points.

Due to a typo, the surface $x^2 = 2y - y^2$ appeared as $x = 2y - y^2$ in the printed question which then requires the solution of a fourth degree polynomial whose roots are non-trivial.

NAME: STUDENT NO:

**Q-3)** Use Stokes' theorem to evaluate the integral $\displaystyle\int_C \frac{y^2}{2}\,dx + zdy + xdz$, where $C$ is the curve of intersection of the plane $x+z=1$ and the ellipsoid $x^2+2y^2+z^2=1$, oriented counterclockwise as viewed from above.

**Solution:**

Let $S$ be the planar region contained inside $C$ on the plane $x+z=1$, and set $F = (M, N, P) = (y^2/2,\ z,\ x)$. Then

$$\int_C \frac{y^2}{2}\,dx + z\,dy + x\,dz = \int_C F\cdot dr$$

and Stokes' theorem says

$$\int_C F\cdot dr = \int\int_S \nabla\times F\cdot n\,d\sigma,$$

where $n = (1/\sqrt{2}, 0, 1\sqrt{2})$ is the unit normal vector of $S$ pointing upwards to be compatible with the orientation of $C$.

$d\sigma$ is the area element on the surfaces $S$. Here we can take $f(x,y,z) = x+z-1 = 0$ for $S$. Then $d\sigma = \dfrac{|\nabla f|}{|\nabla f\cdot p|}\,dA$, where $p = (0,0,1)$ is the unit normal vector of the projection of $S$ onto $xy$-plane. This gives $d\sigma = \sqrt{2}\,dA$.

We also have

$$\begin{aligned}
\nabla\times F\cdot n &= (P_y - N_z,\ M_z - P_x,\ N_x - M_y)\cdot n\\
&= (-1,-1,-y)\cdot(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\\
&= -\frac{1}{\sqrt{2}}(1+y).
\end{aligned}$$

Let $D$ be the projection of $S$ onto $xy$-plane. We find its bounding curve by eliminating $z$ from the equations $x+z=1$ and $x^2+2y^2+z^2=1$. This gives the circle $x^2-x+y^2=0$.

Then we have

$$\begin{aligned}
\int\int_S \nabla\times F\cdot n\,d\sigma &= -\int\int_D (1+y)\,dA\\
&= -\int\int_D dA - \int\int_D y\,dA\\
&= -\frac{\pi}{4},
\end{aligned}$$

Where the first integral gives the area of the circle while the second integral is zero since the odd function $y$ is integrated around a symmetrical region around zero.

NAME: STUDENT NO:

**Q-4)** Set up (but don't evaluate) the surface area integral of $y = z^2 + x$ inside $x^2 + y^2 = 1$.

**Solution:** Let $S$ be this surface. It is given by $f(x, y, z) = z^2 + x - y = 0$. We need to express $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA$ in terms of $x$ and $y$, since we will consider projection of $S$ on $xy$-plane.

We have $\nabla f = (1,\, -1,\, 2z)$, $|\nabla f| = \sqrt{2 + 4z^2}$, $|\nabla f \cdot \mathbf{k}| = |2z|$.

After a careful examination of the projection of $S$ on $xy$-plane, we can write

$$\text{Surface Area of } S = \frac{1}{\sqrt{2}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x}^{\sqrt{1-x^2}} \sqrt{2 + \frac{1}{y - x}}\, dy\, dx + \frac{1}{\sqrt{2}} \int_{-1}^{-1/\sqrt{2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2 + \frac{1}{y - x}}\, dy\, dx.$$

NAME: STUDENT NO:

**Q-5)** Let $\mathbf{G} = \dfrac{5}{x^2+y^2+z^2}\,\mathbf{i} + \dfrac{\cos x\,\sin y}{(x^2+y^2+z^2)^5}\,\mathbf{j} + \dfrac{e^x\,e^y\,e^z}{x^4+y^4+z^4}\,\mathbf{k}$. Find the flux of $\mathbf{curl}(\mathbf{G})$ across the surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 16\}$ away from the origin.

**Solution:** Let $S = A \cup B$ where $A$ is the upper hemisphere and $B$ is the lower hemisphere. Let $C$ be the boundary of $A$ correctly oriented. Then the boundary of $B$ is $-C$. By Stokes' theorem the flux of $\mathbf{curl}(\mathbf{G})$ across $A$ away from the origin is the circulation of $\mathbf{G}$ along $C$, and the flux of $\mathbf{curl}(\mathbf{G})$ across $B$ away from the origin is the circulation of $\mathbf{G}$ along $-C$. Adding these up gives zero. In fact the flux of $\mathbf{curl}(\mathbf{G})$ across any closed orientable surface is zero.

Note that here we can not use divergence theorem since $\mathbf{G}$ is not defined everywhere inside $S$.