Date: June 20, 2009, Saturday NAME:..
Time: 10:00-12:00

STUDENT NO:..

SECTION NUMBER: ....

Math 116 Intermediate Calculus III – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit. Without the correct **section number**, your grade may not be entered in SAPS.

Q-1) **(i)** Prove that $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{x^2y\sin x}{x^2+y^2}\right)=0.$

(ii) Show that $\displaystyle\lim_{(x,y)\to(0,0)}\left(\frac{x^2y}{x^4+y^2}\right)$ does not exist.

Solution (Özgün Ünlü): **(i)** First observe that

$$\left|\frac{x^2y\sin x}{x^2+y^2}\right|\le |y|\,|\sin x|\,\frac{x^2}{x^2+y^2}\le |y|\,\frac{x^2}{x^2+y^2}\le |y|.$$

Hence we have

$$-|y|\le \frac{x^2y\sin x}{x^2+y^2}\le |y|.$$

We also have

$$\lim_{(x,y)\to(0,0)} \mp|y|=0.$$

Then use the sandwich theorem to conclude that the limit is zero.

Solution: **(ii)** Try the path $y=\lambda x^2$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\lambda x^2}}\frac{x^2y}{x^4+y^2}=\lim_{x\to0}\frac{\lambda x^4}{x^4+\lambda^2x^4}=\lim_{x\to0}\frac{\lambda}{1+\lambda^2}=\frac{\lambda}{1+\lambda^2}.$$

The limit on different paths do not agree so the limit does not exist.

NAME: STUDENT NO:

Q-2) Let $w = f(u, v)$, where f has continuous first and second order partial derivatives. Let $u = x^2y$ and $v = \dfrac{x}{y}$. Find w_{xy} at the point $(x, y) = (2, 1)$ if $f_u(4, 2) = 4$, $f_v(4, 2) = -5$, $f_{uu}(4, 2) = -1$, $f_{vu}(4, 2) = 3$ and $f_{vv}(4, 2) = 2$.

Solution (Hamza Yeşilyurt): By chain rule, we find that

$$w_x = f_u u_x + f_v v_x.$$

Similarly by applying the chain rule to the functions f_u and f_v, we find that

$$(f_u)_y = (f_u)_u u_y + (f_u)_v v_y = f_{uu} u_y + f_{uv} v_y$$

and

$$(f_v)_y = (f_v)_u u_y + (f_v)_v v_y = f_{vu} u_y + f_{vv} v_y.$$

Therefore,

$$\begin{aligned} w_{xy} &= (f_u u_x + f_v v_x)_y \\ &= (f_u)_y u_x + f_u u_{xy} + (f_v)_y v_x + f_v v_{xy} \\ &= (f_{uu} u_y + f_{uv} v_y) u_x + f_u u_{xy} + (f_{vu} u_y + f_{vv} v_y) v_x + f_v v_{xy}. \end{aligned}$$

Next, $u_x = 2xy$, $u_x(2, 1) = 4$, $u_y = x^2$, $u_y(2, 1) = 4$, $v_x = 1/y$, $v_x(2, 1) = 1$, $v_y = -x/y^2$, $v_y(2, 1) = -2$, $u_{xy} = 2x$, $u_{xy}(2, 1) = 4$, $v_{xy} = -1/y^2, v_{xy}(2, 1) = -1$, and $u(2, 1) = 4, v(2, 1) = 2$. Moreover, $f_{uv}(4, 2) = f_{vu}(4, 2)$ since f has continuous partial derivatives of first and second order. Hence,

$$w_{xy}(2, 1) = \big((-1).4 + 3.(-2)\big).(4) + 4.4 + \big(3.(4) + 2.(-2)\big).(1) + (-5).(-1) = -11.$$

NAME: STUDENT NO:

Q-3) Show that the ellipsoid $3x^2+2y^2+z^2=4$ and the sphere $x^2+y^2+z^2-5x+3z+6=0$ are tangent to each other at $(1,0,-1)$. Find an equation of this common tangent plane.

Solution (Müfit Sezer):

Let $f(x) = 3x^2 + 2y^2 + z^2$ and $g(x) = x^2 + y^2 + z^2 - 5x + 3z + 6$. Then we have $\nabla f = 6x\mathbf{i}+4y\mathbf{j}+2z\mathbf{k}$ and $\nabla g = (2x-5)\mathbf{i}+(2y)\mathbf{j}+(2z+3)\mathbf{k}$. We get $\nabla f(1,0,-1) = 6\mathbf{i}-2\mathbf{k}$ and $\nabla g(1,0,-1) = -3\mathbf{i}+\mathbf{k}$.

Therefore $\nabla f(1,0,-1) = -2\nabla g(1,0,-1)$ and hence the normal vectors to the respective tangent planes are parallel. Since these planes also share a common point, they are the same. The equation of this plane is

$$6(x-1) - 2(z+1) = 0.$$

NAME: STUDENT NO:

Q-4) Find the highest and the lowest points on the ellipse $3x^2 + 2xy + 2y^2 + x - 7y = 7$.

Solution (Sinan Sertöz): One approach to this problem is to use Lagrange Multipliers Method. The function to optimize is $f(x, y) = y$ subject to the constraint $g(x, y) = 3x^2 + 2xy + 2y^2 + x - 7y - 7 = 0$.

$\nabla f = \lambda \nabla g$ gives $6x + 2y + 1 = 0$ (together with $2x + 4y - 7 = 1/\lambda$.)

We can also argue as follows: At the lowest and highest points of the ellipse its tangent line will be horizontal. The gradient ∇g, being perpendicular to the tangent line will be a vertical vector and its x-component will be zero, i.e. $6x + 2y + 1 = 0$.

Once we get the equation $6x + 2y + 1 = 0$, we solve for y and substitute into g.

$$g(x, -\frac{6x + 1}{2}) = 15x^2 + 27x - 3 = 0.$$

This gives $x = \dfrac{-9 \pm \sqrt{101}}{10}$.

Alternatively we could solve for x from $6x + 2y + 1 = 0$ and then substitute into g.

$$g(-\frac{2y + 1}{6}, y) = 20y^2 - 88y - 85 = 0.$$

This then gives $y = \dfrac{22 \pm \sqrt{101}}{10}$.

Once x or y is found, the other value can of course be found from $6x + 2y + 1 = 0$.

This gives two critical points $p = \left(\dfrac{-9 - \sqrt{101}}{10}, \dfrac{22 + \sqrt{101}}{10} \right)$ and $q = \left(\dfrac{-9 + \sqrt{101}}{10}, \dfrac{22 - \sqrt{101}}{10} \right)$.

Comparing the values of f at these points we find that $f(p) > f(q)$, so the highest point is p and the lowest point is q.

Another approach is to consider the restriction of the function y to the ellipse, which then makes it a function of x. Differentiating $g(x, y) = 0$ implicitly with respect to x, solving for y' and equating it to zero gives $6x + 2y + 1 = 0$. Then proceed as above. (Thanks to Abidin Erdem Karagül.)

NAME: STUDENT NO:

Q-5) Find and classify all critical points of $f(x,y) = 4x^3 + xy^2 - 2yx^2 - x$.

Solution (Aydan Pamir): To find the critical points:

$f_x(x,y) = 12x^2 + y^2 - 4xy - 1 = 0$ **(1)**

$f_y(x,y) = 2xy - 2x^2 = 2x(y-x) = 0$ **(2)**

Equation **(2)** $\Rightarrow x = 0$ or $x = y$

$x = 0$ and Equation **(1)** $\Rightarrow y^2 = 1 \Rightarrow y = \pm 1$, i.e, $P_1(0,1)$ and $P_2(0,-1)$

$y = x$ and Equation **(1)** $\Rightarrow 12x^2 + x^2 - 4x^2 - 1 = 0 \Rightarrow x = \pm 1/3$, i.e, $P_3(1/3, 1/3)$ and $P_4(-1/3, -1/3)$

$\Delta = f_{xx}f_{yy} - f_{xy}^2$ where $f_{xx} = 24x - 4y$, $f_{yy} = 2x$ and $f_{xy} = 2y - 4x$

So,

$\Delta = 2x(24x - 4y) - (2y - 4x)^2 = 8x(6x - y) - 4(y - 2x)^2$

- $\Delta|_{P_1(0,1)} = -4 < 0 \Rightarrow f$ has a saddle point at $P_1(0,1)$
- $\Delta|_{P_2(0,-1)} = -4 < 0 \Rightarrow f$ has a saddle point at $P_2(0,-1)$
- $\Delta|_{P_3(1/3,1/3)} = 4 > 0$ and $f_{xx}|_{P_3(1/3,1/3)} = 20/3 > 0 \Rightarrow$

f has a local minimum at $P_3(1/3, 1/3)$

- $\Delta|_{P_4(-1/3,-1/3)} = 4 > 0$ and $f_{xx}|_{P_4(-1/3,-1/3)} = -20/3 < 0 \Rightarrow$

f has a local maximum at $P_4(-1/3, -1/3)$