Math 123 – Homework 2 – Solutions

Q-1) Let S_n be the permutation group on n objects. Show that S_2 is abelian but S_n is not abelian for any $n > 2$.

Solution: $o(S_n) = n!$, so in particular $o(S_2) = 2$ and is abelian since there is only one group, up to isomorphism, of order 2 and it is abelian.

Observe that S_n can be considered as a subgroup of every S_m for any $m > n$; S_n simply permutes the first n elements leaving the rest unchanged. Thus if we can show that there are two elements $a, b \in S_3$ such that $ab \neq ba$, then that will prove that each S_n with $n > 2$ is non-abelian. For this let $a = (123)$ and $b = (12)$. Check that $(123) \circ (12) = (321)$ and $(12) \circ (123) = (132)$, where $\circ$ denotes composition of the permutations as functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$.

Q-2) If G is a group with the property that $(ab)^2 = a^2b^2$ for all $a, b \in G$, then show that G is abelian.

Solution:

$$\begin{aligned} (ab)^2 &= a^2b^2 \\ abab &= aabb \\ a^{-1}(abab)b^{-1} &= a^{-1}(aabb)b^{-1} \\ ba &= ab. \end{aligned}$$

Q-3) Show that in S_3 there are four elements satisfying $x^2 = e$ and three elements satisfying $y^3 = e$.

Solution: The four elements satisfying $x^2 = e$ are e, (12), (13), (23), and the three elements satisfying $y^3 = e$ are e, (123), (132).

Q-4) Let G be a nonempty set closed under an associative product such that there is an element $e \in G$ with the properties that *(i)* $a \cdot e = a$ for all $a \in G$, and *(ii)* for all $a \in G$ there is an element $i(a) \in G$ with $a \cdot i(a) = e$. Show that G is a group with this operation.

Solution: We first show that every right inverse is also a left inverse. Let $a \in G$, and set $i(a) = b$, $i(b) = c$. We have $ab = e$ and $bc = e$. On one hand we have $abc = (ab)c = ec$, and on the other hand we have $abc = a(bc) = ae = a$. Thus $a = ec$. Now $ba = b(ec) = (be)c = bc = e$. Hence every right inverse is also a left inverse.

Next we show that e is also a left identity. Let $a \in G$. Again set $b = i(a)$. We just showed that $ab = ba = e$. Now $ea = (ab)a = a(ba) = ae = a$.

Thus we showed that the requirements for G to be group are satisfied.

Q-5) Let G be a group and H a subgroup. For any $a, b \in G$ define $a \sim b$ if $ab^{-1} \in H$. We say a is congruent to b mod H, and write $a \equiv b \mod H$. Show that this is an equivalence relation.

Solution: For every $a \in G$, $aa^{-1} = e \in H$, so $a \sim a$.

If $a \sim b$, then $ab^{-1} \in H$ so $(ab^{-1})^{-1} = ba^{-1} \in H$, and $b \sim a$.

If $a \sim b$ and $b \sim c$, then $ab^{-1}, bc^{-1} \in H$ so $ab^{-1}bc^{-1} = ac^{-1} \in H$ and $a \sim c$.

Hence this is an equivalence relation.

Q-6) Let G be a group, H a subgroup and $a \in G$ an element. Define the following subsets of G:

$$\begin{aligned} N(a) &= \{x \in G \mid xa = ax \}, \\ N(H) &= \{x \in G \mid xHx^{-1} = H \}, \\ C(H) &= \{x \in G \mid \forall a \in H,\ xa = ax \}, \\ Z &= \{x \in G \mid \forall a \in G,\ xa = ax \}. \end{aligned}$$

Prove that these are subgroups of G. ($N(a)$ and $N(H)$ are called the *normalizer* of a and H in G, respectively. $C(H)$ is called the *centralizer* of H in G. Z is called the *center* of G.)

Solution:

Recall that for H to be a subgroup of G we have to show that (i) if $x, y \in H$ then $xy \in H$, and (ii) if $x \in H$ then $x^{-1} \in H$.

N(a) is a subgroup: Let $x, y \in N(a)$. Then $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$, so $xy \in N(a)$. And $xa = ax$, $x^{-1}(xa)x^{-1} = x^{-1}(ax)x^{-1}$, $ax^{-1} = x^{-1}a$, so $x^{-1} \in N(a)$.

N(H) is a subgroup: Let $x, y \in N(H)$. Then $(xy)H(xy)^{-1} = xyHy^{-1}x^{-1} = xHx^{-1} = H$, and $H = x^{-1}(xHx^{-1})x = x^{-1}Hx$.

C(H) is a subgroup: This is similar to the first part if you notice that $x \in N(H)$ means that x commutes with elements of H.

Z is a subgroup: This is again similar to the first case.

Q-7) Let $\phi : G \to H$ be a homomorphism between the groups G and H. Define the kernel of ϕ as $\ker \phi = \{x \in G \mid \phi(x) = e_H \}$ where e_H is the identity of H. Show that $\ker \phi$ is a normal subgroup of G.

Solution: We want to show that for every $g \in G$ and for every $h \in \ker \phi$, we must have $ghg^{-1} \in \ker \phi$. But since ϕ is a homomorphism we have $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)e_H\phi(g)^{-1} = e_H$, and hence the result.

Grading: Problem 6 is 40 points, the other problems are 10 points each.

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