Math 123 – Homework 2

Due date: 30 December 2008 Tuesday

Please take your homework solutions to room SA144, Ali Adalı's office before 17:00.

Q-1) Let S_n be the permutation group on n objects. Show that S_2 is abelian but S_n is not abelian for any $n > 2$.

Q-2) If G is a group with the property that $(ab)^2 = a^2b^2$ for all $a, b \in G$, then show that G is abelian.

Q-3) Show that in S_3 there are four elements satisfying $x^2 = e$ and three elements satisfying $y^3 = e$.

Q-4) Let G be a nonempty set closed under an associative product such that there is an element $e \in G$ with the properties that *(i)* $a \cdot e = a$ for all $a \in G$, and *(ii)* for all $a \in G$ there is an element $i(a) \in G$ with $a \cdot i(a) = e$. Show that G is a group with this operation.

Q-5) Let G be a group and H a subgroup. For any $a, b \in G$ define $a \sim b$ if $ab^{-1} \in H$. We say a is congruent to b mod H, and write $a \equiv b \mod H$. Show that this is an equivalence relation.

Q-6) Let G be a group, H a subgroup and $a \in G$ an element. Define the following subsets of G:

$$\begin{aligned}
N(a) &= \{x \in G \mid xa = ax\}, \\
N(H) &= \{x \in G \mid xHx^{-1} = H\}, \\
C(H) &= \{x \in G \mid \forall a \in H,\ xa = ax\}, \\
Z &= \{x \in G \mid \forall a \in G,\ xa = ax\}.
\end{aligned}$$

Prove that these are subgroups of G. ($N(a)$ and $N(H)$ are called the *normalizer* of a and H in G, respectively. $C(H)$ is called the *centralizer* of H in G. Z is called the *center* of G.)

Q-7) Let $\phi : G \to H$ be a homomorphism between the groups G and H. Define the kernel of ϕ as $\ker\phi = \{x \in G \mid \phi(x) = e_H\}$ where e_H is the identity of H. Show that $\ker\phi$ is a normal subgroup of G.

Grading: Problem 6 is 40 points, the other problems are 10 points each.

Please forward any comments or questions to `sertoz@bilkent.edu.tr`
