Math 123 – Homework 3 – Solutions

Due date: 7 January 2009 Wednesday

Please take your homework solutions to room SA144, Ali Adalı's office before 17:00.

Q-1) For a finite group G, show that if $o(G)$ is even, then there is a non-trivial element $a \in G$ such that $a^{-1} = a$.

Solution: Assume not. The inverse of every non-trivial element is non-trivial and every non-trivial element can be paired up with its inverse and this gives the count of non-trivial elements as even. Together with the trivial element e, the order of G becomes odd, a contradiction.

Q-2) Let $\phi : G \to H$ be a group homomorphism. Show that ϕ is one-to-one if and only if $\ker \phi = \{e\}$.

Solution: Let $a, b \in G$ be such that $\phi(a) = \phi(b)$. Then $\phi(a)\phi(b)^{-1} = e_H$, $\phi(ab^{-1}) = e_H$ and $ab^{-1} \in \ker \phi$.

If $\ker \phi = \{e\}$, then $ab^{-1} = e$, $a = b$ and ϕ is one-to-one.

If ϕ is one-to-one, then the only element mapping to e_H is e, so $\ker \phi = \{e\}$.

Q-3) Let $\phi : G \to H$ be a group homomorphism. Show that $\phi(G)$ is a subgroup of H and is isomorphic to the quotient group $G/\ker \phi$

Solution: That $\phi(G)$ is a subgroup follows from the fact that ϕ is a group homomorphism. For example if $\phi(a), \phi(b) \in \phi(G)$, then $\phi(a)\phi(b) = \phi(ab) \in \phi(G)$.

We know that $\ker \phi$ is a normal subgroup. The quotient $G/\ker \phi$ is then the group of right cosets of $\ker \phi$ in G.

For notational convenience set $K = \ker \phi$.

Define $\alpha : G/K \to \phi(G)$ by the rule $\alpha(Ka) = \phi(a)$. This map is well defined. In other words let another representative be used for the coset Ka, for example let $Ka = Kb$. Then $ab^{-1} \in K$, $e_H = \phi(ab^{-1}) = \phi(a)\phi(b)^{-1}$, $\phi(a) = \phi(b)$.

Clearly α is onto. Let $\alpha(Ka) = \phi(a) = e_h$. Then $a \in K$ and $Ka = K$, so α is also one-to-one, hence an isomorphism.

Q-4) Let $\theta \in S_n$ be a 2-cycle. Show that $\prod_{i<j} (x_i - x_j) = -\prod_{i<j} \left(x_{\theta(i)} - x_{\theta(j)}\right)$.

Solution: First observe that for any $m = 1, 2, \ldots, n-1$,

$$\prod_{i<j} (x_i - x_j) = \left[\prod_{\substack{i<j \\ (i,j)\neq(m,m+1)}} (x_i - x_j) \right] [x_m - x_{m+1}] \,.$$

Now it is easy to check that if θ interchanges two consecutive indices, say $\theta = (m, m+1)$, then the claim holds.

If θ interchanges m and $m+k$, then we can consider it as first interchanging m with the neighbouring indices until m takes the place of $m+1$. It forces k sign changes by the above observation. Now to bring $m+1$ to the original place of m, θ needs $k-1$ switches and forces $k-1$ more sign changes, resulting in a net change in sign, as the claim goes.

Q-5) Let G be a finite group and H a subgroup with the property that $i(H)$ is the smallest prime p dividing the order of G. Show that H is a normal subgroup of G.
Hint: Show that G permutes the set of right cosets of H and that the kernel must be contained in H. Now use Lagrange's theorem together with the fact that no prime larger than or equal to p can divide $(p-1)!$.

Solution: Let K be the set of right cosets of H in G. The cardinality of K is $i(H) = p$. (Here $i(H) = o(G)/o(H)$ and is called the index of H in G.) The symmetric group S_p acts on K by simply permuting its elements. Each element of G also permutes elements of K by simply multiplying each right coset from the right and hence sending it onto another right coset, not necessarily distinct than the original one. This defines a map $\phi : G \to S_p$. Check that this defines a homomorphism. We know that $\phi(G)$ is a subgroup of S_p, so $o(G)$ divides the order of S_p which is $p!$.

If $a \in \ker\phi$. Then a leaves each right coset of H fixed, in particular $H = Ha$, so $a \in H$. Hence $\ker\phi$ is a subgroup of H and its order must divide the order of H. Let $m\, o(\ker\phi) = o(H)$ for some positive integer m.

Since $o(H)|o(G)$, m must also divide the order of G. By our description of p, if q is a prime dividing m, then $q \geq p$.

We know that $\phi(G)$ is isomorphic to $G/\ker\phi$, so $o(\phi(G)) = o(G)/(o(H)/m) = m\, o(G)/o(H) = m\, i(H) = mp$. We know that this number divides $p!$, so $m|(p-1)!$.

If q is a prime dividing m, then $q|(p-1)!$ so q is a prime strictly less than p. This contradicts what we found about q above. So no prime divides m, forcing $m = 1$.

This says that $H = \ker\phi$ and hence is a normal subgroup since all kernels are normal.

Please forward any comments or questions to `sertoz@bilkent.edu.tr`
