Date: October 24, 2008, Friday
Time: 10:40-12:30
Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 123 Abstract Mathematics I – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit. For this exam take $\mathbb{N} = \{1, 2, \dots\}$.

Q-1) For a function $f : \mathbb{R} \to \mathbb{R}$ we have the following property:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Write the negation of the above property.

Solution: The negation of the above property is

$$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists \epsilon > 0, \forall \delta > 0, \ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon.$$

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Q-2) Let p, q, r be some logical statements. Show, using a truth table, that $p \Rightarrow q$ is the same thing as $(\sim p) \vee q$. Show also, using whatever you like, that $(p \Rightarrow q) \Rightarrow r$ is not the same thing as $p \Rightarrow (q \Rightarrow r)$.

Solution:

p	$\sim p$	q	$p \Rightarrow q$	$(\sim p) \vee q$
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	1	1	1
1	0	0	0	1	1	1
0	1	1	1	1	1	1
0	1	0	1	0	**0**	**1**
0	0	1	1	1	1	1
0	0	0	1	1	**0**	**1**

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Q-3) Prove that for all positive integers n, we have

$$1^4 + 2^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

Solution: When $n = 1$, both sides are 1.

Assume the statement for n. Now add $(n+1)^4$ to both sides of the statement and check that, after simplification, $\frac{1}{5}(n+1)^5 + \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n + (n+1)^4$. This proves the statement for all $n \in \mathbb{N}$.

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Q-4) Let S be the set of all finite subsets of $\mathbb{N}$. Show that S is countable.

Solution: Let S_n be the set of all subsets of $\mathbb{N}$ containing exactly n elements. Then $S = \bigcup_{n=1}^{\infty} S_n$. If each S_n is countable, then S being a countable union of countable sets will be countable. So it remains to show that each S_n is countable.

Clearly S_1 is countable being in one-to-one correspondence with $\mathbb{N}$. Assume S_n is countable. We observe that there is an injection $S_{n+1} \to \mathbb{N} \times S_n$ given by $A \in S_{n+1} \mapsto a \times A \setminus \{a\}$ where a is the smallest integer in A. Since we assumed S_n to be countable, the product $\mathbb{N} \times S_n$ is countable. Every subset of a countable set is countable (or finite). So S_{n+1} is countable. This completes the proof that each S_n is countable. And that in turn completes the proof that S is countable.

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Q-5) Let $A = \mathbb{N} \times \mathbb{N} \times \cdots$ (*infinite product*), and let B be the set of all infinite sequences of 0 and 1, i.e. a typical element $b \in B$ looks like $b_1 b_2 b_3 \ldots$ where each b_n is either 0 or 1, $n = 1, 2, \ldots$.

(a) Prove or disprove: A is countable.

(b) Which of the following statements is true? Prove your answer.

(i): $\text{card}(A) < \text{card}(B)$, **(ii):** $\text{card}(A) = \text{card}(B)$, **(iii):** $\text{card}(A) > \text{card}(B)$,

(iv): None, because the cardinalities of A and B cannot be compared.

Solution:

(a) We prove that A is uncountable. This follows from the well-known Cantor diagonal argument as follows. Suppose that A is countable. Then we can number and list all elements of A. A typical element in the list would look like $\mathbf{a_k} = (a_{k1}, a_{k2}, a_{k3}, \cdots)$ where $\mathbf{a_k} \in A$ and $a_{kj} \in \mathbb{N}$, $k, j = 1, 2, \ldots$. Now consider the element $\mathbf{c} = (c_1, c_2, c_3, \cdots) \in A$ constructed as follows:

$$c_k = \begin{cases} 1 & \text{if } \; a_{kk} \neq 1, \\ 2 & \text{if } \; a_{kk} = 1. \end{cases}$$

Then clearly $\mathbf{c}$ is not in the above list since it differs from each $\mathbf{a_k}$ in the k-th entry. This contradicts the assumption that A can be counted.

(b) We show that $\text{card}(A) = \text{card}(B)$.

If $\mathbf{b} = b_1 b_2 b_3 \cdots \in B$, then we can send it injectively to A as $(b_1 + 1, b_2 + 1, b_3 + 1, \cdots)$.

Finding an injection from A to B is a little more tricky: We explain it on an example. Let $\mathbf{a_k} = (a_{k1}, a_{k2}, a_{k3}, \cdots) = (2, 10, 5, 4, 1, \cdots) \in A$. We expand each a_{kj} in $\mathbf{a}$ in binary form and write it downwards in reverse order under a_{kj} and fill the remaining columns downward with zeros.

2	10	5	4	1	$\cdots$
0	0	1	0	1	$\cdots$
1	1	0	0	0	$\cdots$
0	0	1	1	0	$\cdots$
0	1	0	0	0	$\cdots$
0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Then we construct a binary sequence from this table by listing the elements in the diagonals, always starting from the left hand side of the table and going upward:
0 10 011 0000 01101 ... Here we left the blanks only to describe the method. In actuality no blanks are needed. This clearly gives an injection of A into B. We now invoke the Schroeder-Bernstein theorem and conclude that $\text{card}(A) = \text{card}(B)$.