Date: December 19, 2008, Friday NAME:..
Time: 10:40-12:30
Ali Sinan Sertöz STUDENT NO:..

Math 123 Abstract Mathematics I – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. A correct answer without proper reasoning may not get any credit.

Q-1) For a collection U of subsets of $\mathbb{N}$, we have the following property:

$$\forall\, x \in U,\ \exists\, y \subset x,\ \text{such that } \mathbf{card}(y) < \infty \text{ and } \sum_{n\in y} n \geq 0.$$

Write the negation of the above property.

Solution: The negation of the above property is

$$\exists\, x \in U,\ \text{such that } \forall\, y \subset x,\ \text{either } \mathbf{card}(y) = \infty \ \text{ or } \sum_{n\in y} n < 0.$$

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Q-2) Find a polynomial formula, and prove it, for the sum

$$S(n) = 1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1) \cdot (2n),\ n \in \mathbb{N}.$$

Solution:

For every new k, we add the term $(2k-1) \cdot (2k) = 4k^2 - 2k$. So up to n, we have 4 times the sum of squares minus 2 times the sum of integers. This gives

$$S(n) = 4 \cdot \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n(n+1)}{2} = \frac{1}{3}\left(n(n+1)(4n-1)\right),$$

which can now be easily proved by induction.

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Q-3) Show that $\frac{1}{12}$ is a point of the Cantor set.

Solution: $12 = 9 + 3 = (110)_3$. Using long division in base 3, calculate the reciprocal of $(110)_3$ to find

$$\frac{(1)_3}{(110)_3} = 0._3 002020202\ldots$$

In fact, check that

$$\frac{2}{3^3} + \frac{2}{3^5} + \frac{2}{3^7} + \cdots \frac{2}{3^{2n+3}} + \cdots = \frac{2}{27}\left(1 + \frac{1}{9} + \cdots + \frac{1}{9^n} + \cdots\right) = \frac{1}{12}.$$

Since the ternary expansion of $1/12$ consists of only 0s and 2s, it belongs to the Cantor set.

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Q-4) Let $P(A)$ denote the power set, i.e. the set of all subsets of the set A. Show that for a non-empty set A, the cardinality of $P(A)$ is always strictly greater than that of A.

Solution: Here I take the solution verbatim from the textbook.

First observe that the function $f : A \to P(A)$ defined as $f(a) = \{a\}$ is one-to-one. Thus we see that the cardinality of A is $\leq$ the cardinality of $P(A)$. We need to show that there is no function from A *onto* $P(A)$. Assume $g : A \to P(A)$ is an onto function. Define $B = \{a \in A \mid a \notin g(a)\ \}$. (The fact that you can choose such a function is due to the axiom of choice!) Since g is onto, there is an element $z \in A$ such that $g(z) = B$. But due to the definition of B, we have $z \in B$ if and only if $z \notin g(z) = B$. This contradiction shows that no such g can exist. Therefore, the cardinality of any non-empty set is strictly smaller then the cardinality of its power set.

NAME: STUDENT NO:

Q-5) Find $a, b \in \mathbb{R}$ such that $(1-i)^{2009} = a + ib$, where i is the imaginary number satisfying $i^2 = -1$.

Solution:

$$\begin{aligned}
(1-i)^{2009} &= \left(\sqrt{2}e^{-i\pi/4}\right)^{2009} \\
&= 2^{1004+1/2}e^{-i\pi(502+1/4)} \\
&= 2^{1004}\left(\sqrt{2}e^{-\pi i/4}\right) \\
&= 2^{1004}(1-i).
\end{aligned}$$

Hence $a = -b = 2^{1008}$.