Date: March 28, 2008, Saturday NAME:...
Time: 10:000-12:00
Ali Sinan Sertöz STUDENT NO:...

Math 124 Abstract Mathematics II – Midterm Exam I – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let L_1 and L_2 be two lines in $\mathbb{R}^2$ meeting at a point P with an acute angle of θ. Show that a reflection around L_1, followed by a reflection around L_2 is a rotation. Also find the angle of this rotation.

Solution: Let Q be an arbitrary angle in the plane. Denote the angle QP makes with L_1 by a. Choose your orientation such that $a > 0$. Reflecting Q around L_1 obtain the point Q'. Now the angle $Q'P$ makes with L_2, which we denote by b may be negative with respect to your chosen orientation. This depends on the relative positions of Q, L_1 and L_2. Now standard arguments show that the total effect of these two reflections is a rotation around P with an angle of 2θ.

The important part in this easy exercise is to be aware of all possible cases some of which give negative b as opposed to the obvious geometric figure one is tempted to draw at first impulse.

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Q-2) Let S^2 be the unit sphere in $\mathbb{R}^3$ centered at the origin O, and let $d(-,-)$ denote the spherical metric. Consider the spherical triangle ΔPQR on S^2 where $d(P,Q) = \beta$, $d(P,R) = \gamma$, $d(R,Q) = \alpha$ and the dihedral angle between the planes of POQ and POR is a. Assume without loss of generality that $P = (1,0,0)$ and $Q = (?,0,?)$.
(i) Find the coordinates of Q and R.
(ii) Prove the identity $\cos\alpha = \cos\beta\,\cos\gamma + \sin\beta\,\sin\gamma\,\cos a$.
(iii) Show that $\alpha \leq \beta + \gamma$.

Solution: Since Q lies in the xz-plane and OQ makes an angle of β with $OP = (1,0,0)$, the coordinates of Q are easily seen to be

$$Q = (\cos\beta,\ 0,\ \sin\beta).$$

To find the coordinates of R, for the sake of drawing a figure, assume that R lies in the first octant. Drop perpendiculars from R to all three coordinate planes and mark the known angles. An easy trigonometry will give

$$R = (\cos\gamma,\ \sin\gamma\sin a,\ \sin\gamma\cos a).$$

We note that $|OQ| = OR| = 1$ and the angle between them is α. Hence $\vec{OQ}\cdot\vec{OR} = \cos\alpha$, giving us

$$\cos\alpha = \cos\beta\,\cos\gamma + \sin\beta\,\sin\gamma\,\cos a.$$

Now we observe that $-1 \leq \cos a \leq 1$, so

$$\cos\alpha \geq \cos\beta\,\cos\gamma - \sin\beta\,\sin\gamma = \cos(\beta+\gamma).$$

Since cos is a decreasing function, we finally get

$$\alpha \leq \beta + \gamma.$$

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Q-3) Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive definite symmetric bilinear form.
(i) Show that $|\phi(u,v)|^2 \leq \phi(u,u)\,\phi(v,v)$ for all $u, v \in \mathbb{R}^n$.

(ii) Show that $d(u,v) = \sqrt{\phi(u-v,u-v)}$ defines a metric on $\mathbb{R}^n$.

Solution: The first part is the famous Cauchy-Schwarz inequality for which numerous proofs are available. Here is one.

For any $u, v \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$, we must have

$$\phi(\lambda u + v, \lambda u + v) \geq 0$$

since ϕ is positive definite. We can now expand the left hand side using bilinearity of ϕ to obtain

$$[\phi(u,u)]\lambda^2 + [2\phi(u,v)]\lambda + [\phi(v,v)] \geq 0.$$

Since this holds for all $\lambda \in \mathbb{R}$, the discriminant of the quadratic expression must be non-positive, which in turn is equivalent to

$$|\phi(u,v)|^2 \leq \phi(u,u)\,\phi(v,v).$$

It is clear that $d(u,v) \geq 0$ with equality holding if and only if $u = v$. It is equally clear that $d(u,v) = d(v,u)$. It remains to establish the triangle inequality.

Let $u, v, w \in \mathbb{R}^n$. Then

$$\begin{aligned}
(d(u,w) + d(w,v)) &= \phi(u-w,u-w) + 2\sqrt{\phi(u-w,u-w)\phi(w-v,w-v)} + \phi(w-v,w-v) \\
&\geq \phi(u-w,u-w) + 2\phi(u-w,w-v) + \phi(w-v,w-v) \\
&= \phi(u,u) - 2\phi(u,v) + \phi(v,v) \\
&= \phi(u-v,u-v) \\
&= d(u,v),
\end{aligned}$$

where in the second line we used the Cauchy-Schwarz inequality and in the following lines we used the bilinearity of ϕ. This establishes the triangle inequality.

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Q-4) Let $\mathcal{H}^2$ be the hyperbolic plane in the (t, x, y) space given by the equation $-t^2 + x^2 + y^2 = -1$. Let L_1 and L_2 be two lines in $\mathcal{H}^2$. In each of the following cases decide if a common perpendicular exists for the lines L_1 and L_2. When it exists describe how to construct it.
(i) L_1 and L_2 are diverging.
(ii) L_1 and L_2 are ultraparallel.
(iii) L_1 and L_2 are intersecting.

Solution: Let $L_1 = \pi_1 \cap \mathcal{H}^2$ and $L_2 = \pi_2 \cap \mathcal{H}^2$ where π_1 and π_2 are two planes in $\mathbb{R}^3$ passing through the origin.

Let $V = \pi_1 \cap \pi_2$ be the line of intersection of these two planes and let π be the plane through the origin in $\mathbb{R}^3$ perpendicular to the line V.

If $L = \pi \cap \mathcal{H}^2 \neq \emptyset$, then L is the line in $\mathcal{H}^2$ orthogonal to both L_1 and L_2.

In the given three cases we check if $L = \emptyset$ or not.

(i) If L_1 and L_2 diverge, then V is outside the light cone so π intersects $\mathcal{H}^2$ and L is the required common perpendicular.

(ii) In this case the line V is on the light cone and hence π intersects the light cone along a line which does not intersect $\mathcal{H}^2$. But in this case, since π touches the light cone, we say that L is perpendicular to L_1 and L_2 at infinity, to summarize this relative positions of the lines involved.

(iii) In this case the line V intersects $\mathcal{H}^2$, at the point where L_1 and L_2 intersect each other. Then π does not intersect $\mathcal{H}^2$.