Date: April 25, 2009, Saturday NAME:..
Time: 10:00-12:00
Ali Sinan Sertöz STUDENT NO:..

Math 124 Abstract Mathematics II – Midterm Exam II – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 4 questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. After the exam check the course web page for solutions.

Q-1) Let L_1 and L_2 be two parallel lines in $\mathbb{R}^2$. Explain what it means for these two lines to meet at infinity. Using coordinates of your choice show exactly where they meet.

Solution: Two parallel lines L_1 and L_2 in $\mathbb{R}^2$ are given by $aX + bY + c_i = 0$ where $(a, b) \neq (0, 0)$, and $c_1 \neq c_2$. Assume without loss of generality that $a \neq 0$. Consider the embedding of $\mathbb{R}^2$ into $\mathbb{P}^2$ by the map $(X, Y) \to [X : Y : 1]$ where $[x : y : z]$ are homogeneous coordinates in $\mathbb{P}^2$. Then the images of these parallel lines satisfy the homogeneous equations $ax + by + c_i z = 0$, $i = 1, 2$. Let $[s : t] \in \mathbb{P}^1$ with $0 = [0 : 1]$ and $\infty = [1 : 0]$. Then these lines can be parameterized in $\mathbb{P}^2$ as $[s : t] \to [tc_i - sb : sa : -ta]$. They meet when $t = 0$, which is the point $[-b : a : 0]$ and corresponds to a point on the line at infinity in $\mathbb{P}^2$ with respect to the chart we chose. Notice that this points represents the common slope of the parallel lines.

The key ingredients of this answer are:
(i) writing equation of parallel lines in affine plane.
(ii) projective closure of the affine plane in the projective plane.
(iii) parameterizing a line by $\mathbb{P}^1$.
(iv) recognizing the line at infinity and the points on it.
(v) interpreting the point of intersection as the common slope.

NAME: STUDENT NO:

Q-2) Define the cross-ratio of four distinct and ordered numbers z_1, z_2, z_3, z_4 as the image of z_4 under the unique linear fractional transformation which sends z_1, z_2, z_3 to $0, \infty, 1$ respectively, and denote it by $\langle z_1, z_2, z_3, z_4 \rangle$.

(i) Find the cross-ratio $\langle 1, 2, 3, 4 \rangle$.

(ii) Prove or disprove that there exists a linear fractional transformation $T(z) = \dfrac{az+b}{cz+d}$ such that $\langle 1, 2, 3, 4 \rangle \neq \langle T(1), T(2), T(3), T(4) \rangle$.

Solution: Observe that $\langle z_1, z_2, z_3, z \rangle = \dfrac{z - z_1}{z - z_2} \cdot \dfrac{z_3 - z_2}{z_3 - z_1}$.

Then $\langle 1, 2, 3, 4 \rangle = \dfrac{4-1}{4-2} \cdot \dfrac{3-2}{3-1} = \dfrac{3}{4}$.

Let ϕ be the unique linear fractional transformation sending z_1, z_2, z_3 to $0, \infty, 1$ respectively. Then $\langle z_1, z_2, z_3, z_4 \rangle = \phi(z_4)$ by definition of cross-ratio. Similarly let ψ be the unique linear fractional transformation sending $T(z_1), T(z_2), T(z_3)$ to $0, \infty, 1$ respectively. Then $\langle T(z_1), T(z_2), T(z_3), T(z_4) \rangle = \psi(T(z_4))$ again by definition of cross-ratio.

Now observe that $\psi \circ T$ is a linear fractional transformation sending z_1, z_2, z_3 to $0, \infty, 1$ respectively. By uniqueness of ϕ we must have $\psi \circ T = \phi$. This gives $\psi(T(z_4)) = \phi(z_4)$, or equivalently $\langle z_1, z_2, z_3, z_4 \rangle = \langle T(z_1), T(z_2), T(z_3), T(z_4) \rangle$. Thus the cross-ratio is invariant under linear fractional transformations, and the above statement given in the problem is false.

NAME: STUDENT NO:

Q-3) Let r, k, n be positive integers satisfying $0 < r < k < n$. Let $G(k,n)$ be the space of all k-dimensional vector subspaces of $\mathbb{R}^n$, and for a fixed r-dimensional vector subspace V_r of $\mathbb{R}^n$ define $G(V_r)$ as the space of all k-dimensional vector subspaces of $\mathbb{R}^n$ containing V_r. Notice that $G(V_r) \subset G(k,n)$.
(i) Find $\dim G(k,n)$.
(ii) Find $\dim G(V_r)$.

Solution: Any $V \in G(k,n)$ is spanned by k linearly independent vectors, entries of which form a $k \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}.$$

Since these k rows are linearly independent, there exist k columns which form an invertible matrix. Assume without loss of generality that these are the first k columns. Then multiplying the above matrix with the inverse of the matrix formed by the first k columns gives

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1\,n-k} \\ 0 & 1 & \cdots & 0 & b_{21} & \cdots & b_{2\,n-k} \\ \vdots & & \ddots & & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{k1} & \cdots & b_{k\,n-k} \end{pmatrix}.$$

This shows that there are $k(n-k)$ free parameters, hence $\dim G(k,n) = k(n-k)$.

As for the dimension of $G(V_r)$, write $\mathbb{R}^n = V_r \bigoplus V_R^\perp$, where $V_r^\perp$ is the orthogonal complement of V_r in $\mathbb{R}^n$. Note that $\dim V_r^\perp = n - r$. Let U be any $(k-r)$-dimensional vector subspace of $V_r^\perp$. If we define $V = V_r \bigoplus U$, then clearly $V \in G(V_r)$. Conversely for any $V \in G(V_r)$, write $V = V_r \bigoplus U$ where U is the orthogonal complement of V_r in V. Then clearly U is a $(k-r)$-dimensional vector subspace of $V_r^\perp$.

Thus there is a one-to-one correspondence between the elements of $G(V_r)$ and the $(k-r)$-dimensional subspaces of the $(n-r)$-dimensional vector space $V_r^\perp$. We just proved in the first part that this latter space has dimension $(k-r)[(n-r)-(k-r)] = (k-r)(n-k)$, which is now the dimension of $G(V_r)$.

NAME: STUDENT NO:

Q-4) Prove or disprove: For any finite subset A of $\mathbb{P}^n$ we can find a hyperplane H of $\mathbb{P}^n$ such that $H \cap A = \emptyset$.

Solution: We will construct this hyperplane in $n-1$ steps.

Step 0: Let V_0 be a point in $\mathbb{P}^n$ disjoint from A. If $n = 1$, then $H = V_0$. If $n > 1$, then apply Step 1.

Step k, for $k > 0$, is described as follows:

Assume that we have constructed linear subspaces $V_0 \subset V_1 \subset \cdots \subset V_{k-1}$ of $\mathbb{P}^n$ such that $V_{k-1} \cap A = \emptyset$, and $n > k$. Then we have

Step k: Let V_k be a linear subspace of $\mathbb{P}^n$, containing V_{k-1} and different than any of the finitely many linear subspaces $\mathbf{span}\{V_{k-1}, p\}$, where $p \in A$. It is possible to make this choice since the dimension of the space of all k-linear subspaces of $\mathbb{P}^n$ containing V_{k-1}, which is the dimension of the space of all $(k+1)$-dimensional vector subspaces of $\mathbb{R}^{n+1}$ containing a fixed k-dimensional vector subspace, is $n - k > 0$. Hence there are infinitely many k-dimensional linear subspaces of $\mathbb{P}^n$ containing V_{k-1} and only finitely many of them are *bad*. If $n = k + 1$, then $H = V_k$. If $n > k + 1$, then apply Step $k + 1$.

This process clearly stops and produces the required H, since n is fixed and is finite.