Due Date: 4 January 2016 Monday
Time: 18:30-20:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 202 Complex Analysis – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

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Q-1) Find the images of the circles

$$C_n:\ (x-n)^2+y^2=n^2, \quad \text{and} \quad D_n:\ (x-\frac{1}{n})^2+y^2=\frac{1}{n^2}, \quad \text{where} \quad n=1,2,\ldots.$$

under the Mobiüs transformation $T(z)=\dfrac{1}{z}$.

Solution:

A Mobiüs transformation sends circles to circles, in the extended sense. The circles C_n and D_n pass through the origin and T sends the origin to infinity. Hence the images are circles through infinity, which means that the images are lines. The circle C_n intersects x-axis also at the point $(2n,0)$, and this point is sent by T to the point $(1/(2n),0)$. Hence C_n is sent to the vertical line passing through $(1/(2n),0)$. Similarly D_n is sent to the vertical line passing through $(n/2,0)$. We know that these lines will be perpendicular to the x-axes since the circles intersect x-axes perpendicularly, and the Mobius transformation $z\mapsto 1/z$ is conformal.

We can also algebraically reach to the same conclusion as follows.

$$z\mapsto\frac{1}{z}\quad\text{is the same as}\quad (x,y)\mapsto\left(\frac{x}{x^2+y^2},-\frac{y}{x^2+y^2}\right).$$

Replacing each x and y in $(x-n)^2+y^2=n^2$ with its corresponding image and simplifying we get $x=\dfrac{1}{2n}$. This describes a line perpendicular to x-axes at the point $(1/(2n),0)$, as we found above as the image of C_n. Because of the minus sign in front of the y component of the image, the counterclockwise orientation of C_n becomes downward orientation on the line $T(C_n)$.

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Q-2) Find the principal value of $\alpha = (1+i)^{1+\sqrt{3}\,i}$. Write your answer in rectangular form as $\alpha = A + iB$ where A and B are real numbers. What is $|\alpha|$?

Solution:

$\alpha = \exp(\log \alpha) = \exp((1+\sqrt{3}\,i)\log(1+i)$. Since $(1+i) = \sqrt{2}\exp(i\pi/4)$, we have $\log(1+i) = \ln\sqrt{2}+i\pi/4$. Hence

$$\log \alpha = (1+\sqrt{3}\,i)(\ln\sqrt{2}+i\pi/4) = [\ln\sqrt{2} - \sqrt{3}\pi/4] + i[\sqrt{3}\ln\sqrt{2} + \pi/4] = a + ib.$$

Finally

$$\alpha = e^{a+ib} = e^a \cos b + ie^a \sin b.$$

Now it is clear that $|\alpha| = e^a \approx .3628463059$.

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Q-3) Let $f(z) = \dfrac{ze^{1/z}}{1+z}$.

(a) Find the residue of f at $z = -1$.

(b) Find the residue of f at $z = 0$.

(c) Find the residue of f at $z = \infty$.

Solution:

Since $z = 0$ is an essential singularity, we need to find the Laurent expansion of f at $z = 0$ to determine the residue. In fact we only need to find the coefficient of $1/z$. For this note that for $|z| < 1$ we have

$$f(z) = \frac{z}{1+z}\, e^{1/z} = \left(z - z^2 + z^3 - z^4 + \cdots + (-1)^{n+1} z^n + \cdots\right)\left(1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots + \frac{1}{n! z^n} + \cdots\right).$$

Multiplying these two series but keeping record of only the coefficient of $1/z$ we find that

$$\operatorname{Res}(f(z), 0) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}.$$

On the other hand $z = -1$ is a simple pole and hence

$$\operatorname{Res}(f(z), -1) = ze^{1/z}\Big|_{z=-1} = -\frac{1}{e}.$$

Finally, since the sum of all residues, including the one at infinity is zero, and since the above two residues already add up to zero, we conclude that

$$\operatorname{Res}(f(z), \infty) = 0.$$

If you want to calculate this residue directly, you need the followingarguments.

$$\frac{1}{z^2}\, f(1/z) = \frac{e^z}{z^2(1+z)} = \frac{g(0)}{z^2} + \frac{g'(0)}{z} + \cdots, \quad \text{where} \quad g(z) = \frac{e^z}{1+z}.$$

Note that $g'(0) = 0$. Finally we have

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\Big(\frac{f(1/z)}{z^2}, 0\Big) = 0.$$

In fact

$$\frac{f(1/z)}{z^2} = \frac{1}{z^2} + \frac{1}{2} - \frac{1}{3}\, z + \frac{3}{8}\, z^2 - \frac{11}{30}\, z^3 + \frac{53}{144}\, z^4 - \frac{103}{280}\, z^5 + \frac{2119}{5760}\, z^6 - \frac{16687}{45360}\, z^7 + \cdots,$$

from which it is clear that the residue is zero.

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Q-4) Evaluate the improper real integral

$$\int_0^\infty \frac{x^2}{1+x^4}\, dx.$$

Solution:

Let $f(z) = \frac{z^2}{1+z^4}$. Following the standard arguments in such problems, see for example Example 9.31 on page 315 of the textbook, we conclude that the integral is half the sum of the residues of $f(z)$ in the upper half plane, multiplied by $2\pi i$. The Singularities of $f(z)$ in the upper half plane are

$$a = e^{\pi i/4}, \quad \text{and} \quad b = e^{3\pi i/4}.$$

Since the singularities at these points are simple poles, the residues are calculated as follows.

$$\text{Res}(\frac{z^2}{1+z^4}, a) = \left.\frac{z^2}{4z^3}\right|_{z=a} = \frac{e^{-\pi i/4}}{4} = \frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}.$$

Similarly

$$\text{Res}(\frac{z^2}{1+z^4}, b) = \left.\frac{z^2}{4z^3}\right|_{z=b} = \frac{e^{-3\pi i/4}}{4} = -\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}.$$

Finally

$$\int_0^\infty \frac{x^2}{1+x^4}\, dx = \left(\frac{1}{2}\right)(2\pi i)\left(\text{Res}(\frac{z^2}{1+z^4}, a) + \text{Res}(\frac{z^2}{1+z^4}, b)\right) = \frac{\pi}{2\sqrt{2}}.$$

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Q-5) In your personal opinion, what was the most striking idea that you learned from Complex Analysis? Why was it so striking?
Note that you are not required to prove anything. Just explain in your own words which aspect of Complex Analysis enriched and surprised you most. Don't worry about grading; I will worry about it myself while grading! ☺

Solution:

Anyone who answers this question sincerely gets full marks. That being aside, let's talk business.

Probably the most striking feature of Complex Analysis is the residue theory. It gives a method of evaluating integrals without looking for antiderivatives.

The residue theory on the other hand follows from the theorem of Cauchy which says that the integral of an analytic function around a closed contour is zero. Of course the most spectacular aspect of this theorem is Goursat's proof which does not use the continuity of the derivatives.