Due Date: 21 September 2015, Monday
Time: Class time
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 202 Complex Analysis – Homework 1 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit. **Submit your solutions on this booklet only. Use extra pages if necessary.**

Rules for Homework and Take-Home Exams

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends.

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

Affidavit of compliance with the above rules: I affirm that I have complied with the above rules in preparing this submitted work.

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Q-1) Find a complex form for the hyperbola with the real equation $9x^2 - 4y^2 = 36$.

Solution:

The hyperbola with the canonical equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is the locus of points whose distances from $(c, 0)$ and $(-c, 0)$ differ by $\pm 2a$, where $a, b, c > 0$ and $a^2 + b^2 = c^2$. We can write this in complex notation as

$$|z - c| - |z + c| = \pm 2a, \quad \text{where} \quad z = x + iy.$$

In our case $a = 2$, $b = 3$ and hence $c = \sqrt{13}$.

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Q-2) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers, where n is a fixed positive integer. Prove that

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right).$$

Also find all cases when the equality holds.

Solution:

Define the usual Hermitian inner product on $\mathbb{C}^n$ as follows. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then define

$$\langle x, y\rangle = x_1\bar{y_1} + \cdots + x_n\bar{y_n}.$$

Also define the associated norm as

$$\|x\| = \sqrt{\langle x, x\rangle},$$

where we take the positive square root.

Now let

$$u = (a_1, \ldots, a_n), \quad \text{and} \quad v = (\bar{b_1}, \ldots, \bar{b_n}),$$

where the bar denotes the complex conjugate. Note that

$$\|u\|^2 = \sum_{k=1}^{n} |a_k|^2, \quad \text{and} \quad \|v\|^2 = \sum_{k=1}^{n} |b_k|^2.$$

Therefore we want to show that

$$|\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2.$$

First note that if either u or v is the zero vector, then the inequality holds trivially, both sides being zero. So assume that neither is zero. Let

$$\lambda = \frac{\langle u, v\rangle}{\|v\|^2}, \quad \text{and define the vector} \quad z = u - \lambda v.$$

Note that $\langle v, z\rangle = 0$. Hence from $u = \lambda v + z$ we get

$$\|u\|^2 = |\lambda|^2\|v\|^2 + \|z\|^2 \geq |\lambda|^2\|v\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Multiplying all sides by $\|v\|^2$ we get the required inequality. Note that the equality hold whenever $z = 0$, or equivalently when u and v are linearly dependent over $\mathbb{C}$.

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Q-3) Find all positive integers n such that

$$(1+i)^{2n} = (1+i\sqrt{3})^n = 2^n.$$

Solution: Note that we have $(1+i)^{2n} = 2^n \exp(2i\pi n/4)$. In order to have $\exp(2i\pi n/4) = 1$, we must have $4|n$. Similarly $(1+i\sqrt{3})^n = 2^n \exp(\pi i n/3)$, and we must have $6|n$. So all multiples of 12 will work.

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Q-4) Give an example of a sequence which has the positive integers as its limit points but none of the sequence terms are integers..

Solution:

We can define the terms of the sequence in terms of blocks.
First block: 1.01
Second block: 1.001, 2.001
Third block: 1.0001, 2.0001, 3.0001
n-th block: $1.\underbrace{0\cdots0}_{n-\text{zeros}}1, 2.\underbrace{0\cdots0}_{n-\text{zeros}}1, \ldots, n.\underbrace{0\cdots0}_{n-\text{zeros}}1,$

It is now clear that for each integer n, there is a subsequence converging to n, and none of the terms is an integer itself.

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Q-5) Let $\{S_n\}$ be a sequence of non-empty compact subsets of $\mathbb{C}$ such that $S_{n+1} \subset S_n$ for every $n = 1, 2, \ldots$.. Prove or disprove that $\bigcap_{n=1}^{\infty} S_n = \emptyset$ is possible.

Solution:

This is not possible. Define a sequence by choosing $z_n \in S_n$. This sequence is bounded since all elements lie in S_1 which is bounded. Being bounded and infinite, this sequence has at least one accumulation point. Let z_0 be an accumulation point of this sequence. Note that z_0 is still an accumulation point of the sequence z_n after we truncate finitely many terms. In particular the sequence $z_n, z_{n+1}, \ldots$ lies in S_n, and since S_n is compact, the accumulation point z_0 is in S_n, for every n. Hence $z_0 \in \bigcap_{n=1}^{\infty} S_n$.