Due Date: 21 December 2015, Monday
Time: Class Time
Instructor: Ali Sinan Sertöz

NAME:..................................................

STUDENT NO:.......................................

# Math 202 Complex Analysis – Homework 5 – Solutions

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |
| 20 | 20 | 20 | 20 | 20 | 100 |

*Please do not write anything inside the above boxes!*

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.
**Submit your solutions on this booklet only. Use extra pages if necessary.**

---

## Rules for Homework and Take-Home Exams

(1) You may discuss the problems only with your classmates or with me. In particular you may not ask your assigned questions or any related question to online forums.

(2) You may use any written source be it printed or online. Google search is perfectly acceptable.

(3) It is absolutely mandatory that you write your answers alone. Any similarity with your written words and any other solution or any other source that I happen to know is a direct violation of honesty.

(4) You must obey the usual rues of attribution: all sources you use must be explicitly cited in such a manner that the source is easily retrieved with your citation. This includes any ideas you borrowed from your friends.

(5) Even if you find a solution online, you must rewrite it in your own narration, fill in the blanks if any, making sure that you exhibit your total understanding of the ideas involved.

**Affidavit of compliance with the above rules:** I affirm that I have complied with the above rules in preparing this submitted work.

*Please sign here:*

NAME: STUDENT NO:

**Q-1)** Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{x^4+6x^2+13}\,dx.$

**Solution:**

let $f(z) = \dfrac{x^2}{x^4+6x^2+13}$. The poles of $f$ are at $\pm a$ and $\pm b$ where $a = \sqrt{-3+2i}$ and $b = \sqrt{-3-2i}$, chosen such that both lie in the upper half plane. Setting $a = x+iy$ with $x, y > 0$, we must have $b = -x+iy$. Using the fact that $a^2 = -3+2i$ we find that

$$a - b = 2x = 2\left(\frac{-3+\sqrt{13}}{2}\right)^{1/2}.$$

Since the poles are simple poles, we have

$$\mathrm{Res}_{z=a}\, f(z) = \left.\frac{z^2}{4z^3+12z}\right|_{z=a} = \frac{a}{4a^2+12} = \frac{a}{8i}.$$

Similarly

$$\mathrm{Res}_{z=b}\, f(z) = \left.\frac{z^2}{4z^3+12z}\right|_{z=b} = \frac{b}{4b^2+12} = -\frac{b}{8i}.$$

The value of the integral is equal to

$$\frac{1}{2}\,(2\pi i)\,\left(\mathrm{Res}_{z=a}\, f(z) + \mathrm{Res}_{z=b}\, f(z)\right) = \frac{\pi}{8}(a-b) = \frac{\pi}{8}\left(\frac{-3+\sqrt{13}}{2}\right)^{1/2} \approx 0.432.$$

NAME: STUDENT NO:

**Q-2)** Let $f(z) = \dfrac{z^2}{(z^4+1)^2}$.

(a) Find all poles of $f$.

(b) Show in detail how you calculate the residue of $f$ at one of the poles.

(c) Write all the residues of $f$ at all the poles. Do not show your work.

**Solution:**

The poles are the root of the equation $(z^4+1)=0$. Let $w = e^{\frac{i\pi}{4}} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$. Then all the roots are $w, w^3, w^4, w^7$.

These are double poles since the denominator of $f(z)$ is $(z^4+1)^2$.

$\mathrm{Res}_{z=w} f(z) = \frac{d}{dz}\Big|_{z=w} [(z-w)^2 f(z)]$, which is clear once you consider the Laurent expansion of $f$ at $w$.

$$\mathrm{Res}_{z=w} f(z) = \frac{\sqrt{2}}{32} - i\frac{\sqrt{2}}{32}.$$

$$\mathrm{Res}_{z=w^3} f(z) = -\frac{\sqrt{2}}{32} - i\frac{\sqrt{2}}{32}.$$

$$\mathrm{Res}_{z=w^5} f(z) = -\frac{\sqrt{2}}{32} + i\frac{\sqrt{2}}{32}.$$

$$\mathrm{Res}_{z=w^7} f(z) = \frac{\sqrt{2}}{32} + i\frac{\sqrt{2}}{32}.$$

NAME: STUDENT NO:

**Q-3)** Evaluate the integral $\int_0^\infty \frac{x^2}{(x^4+1)^2}\,dx$.

**Solution:**

After the usual considerations, and also taking into account that the integrand is even, we conclude that the integral is equal to one half of the sum of the residues at poles in the upper half plane multiplied by $2\pi i$. This gives

$$\frac{1}{2}\,(2\pi i)\,\left(\mathrm{Res}_{z=w}\,f(z) + \mathrm{Res}_{z=w^3}\,f(z)\right) = \frac{\pi\sqrt{2}}{16}.$$

NAME: STUDENT NO:

**Q-4)** Evaluate the integral $\displaystyle\int_0^\infty \frac{x\sin x}{x^4+1}\,dx$.

**Solution:**

We integrate $f(z) = \dfrac{ze^{iz}}{z+4+1}$ over a semicircle lying in the upper half plane with center at the origin and radius $R > 1$. This integral is equal to $2\pi i$ times the sum of the residues at $w = e^{\frac{i\pi}{4}}$ and $w^3$.

Taking the limit as $r \to \infty$, the integral along the circular arc goes to zero, while the integral along the real line goe to $2i$ times the integral we want to calculate. This comes from $e^{ix} = \cos x + i\sin x$. Thus

$$\int_0^\infty \frac{x\sin x}{x^4+1}\,dx = \frac{1}{2i}\,(2\pi i)\left(\mathrm{Res}_{z=w} f(z) + \mathrm{Res}_{z=w^3} f(z)\right) = \frac{\pi}{2}\,\frac{\sin\frac{1}{\sqrt{2}}}{e^{\frac{1}{\sqrt{2}}}} \approx 0.5.$$

Here we have

$$\mathrm{Res}_{z=w} f(z) = \frac{1}{4}\frac{\sin\frac{1}{\sqrt{2}}}{e^{\frac{1}{\sqrt{2}}}} - \frac{i}{4}\frac{\cos\frac{1}{\sqrt{2}}}{e^{\frac{1}{\sqrt{2}}}} \quad \text{and} \quad \mathrm{Res}_{z=w^3} f(z) = \frac{1}{4}\frac{\sin\frac{1}{\sqrt{2}}}{e^{\frac{1}{\sqrt{2}}}} + \frac{i}{4}\frac{\cos\frac{1}{\sqrt{2}}}{e^{\frac{1}{\sqrt{2}}}}.$$

NAME: STUDENT NO:

**Q-5)** For any positive integer $n$, define the function

$$f_n(z) = \frac{1}{z}\prod_{k=1}^{n}\frac{z}{1-kz}.$$

Let

$$T_n = \sum_{k=1}^{n} \mathrm{Res}_{z=1/k}\, f(z).$$

Show that

$$T_n = \frac{(-1)^n}{n!}.$$

**Solution:**

The sum of all the residues in $\mathbb{C}$ is equal to the residue at infinity, calculated with the correct sign! We therefore have

$$T_n = \mathrm{Res}_{t=0}\left(\frac{f(1/t)}{t^2}\right) = \mathrm{Res}_{t=0}\,\frac{1}{t(t-1)(t-2)\cdots(t-n)} = \frac{(-1)^n}{n!},$$

as claimed. Here $1/t^2$ comes from $dz$.