Due Date: 21 December 2015 Monday
Time: 17:30-19:30
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 202 Complex Analysis – Makeup Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Find all positive integers n such that

$$(1+i)^{2n} = (1+i\sqrt{3})^n = 2^n.$$

Solution: Note that we have $(1+i)^{2n} = 2^n \exp(2i\pi n/4)$. In order to have $\exp(2i\pi n/4) = 1$, we must have $4|n$. Similarly $(1+i\sqrt{3})^n = 2^n \exp(\pi i n/3)$, and we must have $6|n$. So all multiples of 12 will work.

NAME: STUDENT NO:

Q-2) Calculate all values of $(1+i\sqrt{3})^{(1+i)}$ in rectangular form, i.e. as $A+iB$ with A and B real. Then calculate the modulus of its principal value.

Solution:

We have $(1+i\sqrt{3}) = 2e^{i(\frac{\pi}{3}+2n\pi)}$, where n is any integer.

$$\begin{aligned}
(1+i\sqrt{3})^{(1+i)} &= \exp[(1+i)\log(1+i\sqrt{3})] \\
&= \exp[(1+i)(\ln 2 + i(\frac{\pi}{3}+2n\pi))] \\
&= \exp[(\ln 2 - \frac{\pi}{3} - 2n\pi) + i(\ln 2 + \frac{\pi}{3} + 2n\pi)] \\
&= 2e^{-\frac{\pi}{3}-2n\pi}[\cos(\ln 2 + \frac{\pi}{3} + 2n\pi) + i\sin(\ln 2 + \frac{\pi}{3} + 2n\pi)] \\
&= 2e^{-\frac{\pi}{3}-2n\pi}[\cos(\ln 2 + \frac{\pi}{3}) + i\sin(\ln 2 + \frac{\pi}{3})]
\end{aligned}$$

From this it follows that the principal value, corresponding to $n=0$, of the modulus is

$$\left|(1+\sqrt{3})^{(1+i)}\right| = 2e^{-\frac{\pi}{3}} \approx 0.7.$$

NAME: STUDENT NO:

Q-3) Let S be the square in the plane with vertices at the points $(4,0)$, $(0,4i)$, $(-4,0)$, and $(0,-4i)$. Evaluate the following integral

$$\int_S \frac{z^3}{(z-1)(z-2)(z-3)}\, dz,$$

where S is traced counterclockwise.

Solution:

Using Cauchy integral formula around each of the points $1, 2, 3$, and adding up, we get the following calculation.

$$\int_S \frac{z^3}{(z-1)(z-2)(z-3)}\, dz = 2\pi i \left(\left.\frac{z^3}{(z-2)(z-3)}\right|_{z=1} + \left.\frac{z^3}{(z-1)(z-3)}\right|_{z=2} + \left.\frac{z^3}{(z-1)(z-2)}\right|_{z=3} \right) = 12\pi i.$$

NAME: STUDENT NO:

Q-4) Write the Laurent series of $f(z) = \dfrac{z+4}{1-3z-10z^2}$ converging in the annulus $\dfrac{1}{5} < |z| < \dfrac{1}{2}$.

Solution:

It follows from $\dfrac{1}{5} < |z| < \dfrac{1}{2}$ that

$$|2z| < 1 \quad \text{and} \quad \frac{1}{|5z|} < 1.$$

Then we have

$$f(z) = \frac{1}{1+2z} - \frac{3}{5z}\frac{1}{1-\frac{1}{5z}}.$$

Finally using the geometric series we get

$$f(z) = \sum_{n=0}^{\infty} (-2)^n z^n - \sum_{n=1}^{\infty} \frac{3}{5^n z^n}, \quad \text{for} \quad \frac{1}{5} < |z| < \frac{1}{2}.$$

NAME: STUDENT NO:

Q-5) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^2}{1+x^4}\,dx$.

Solution:

Let $a = e^{i\pi/4}$ and $b = a^3$. The only poles of $\dfrac{z^2}{z^4+1}$ in the upper half plane are a and b, and they are simple poles. Then after all the usual procedures of closed paths and implementation of residue theory, the calculation of the value of the integral boils down to the following.

$$\int_0^\infty \frac{x^2}{1+x^4}\,dx = \frac{1}{2}\,(2\pi i)\left(\frac{a^2}{4a^3} + \frac{b^2}{4b^3}\right) = \frac{\pi\sqrt{2}}{4}.$$