Due Date: 17 October 2015, Saturday
Time: *TBA*
Instructor: Ali Sinan Sertöz

NAME:..

STUDENT NO:..

Math 202 Complex Analysis – Midterm Exam I – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **3** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

NAME: STUDENT NO:

Q-1) Find all $z \in \mathbb{C}$ such that $z^3 = i$. Write your answer in the rectangular form $x + iy$, where x and y are real numbers.

Solution:

We have $z^3 = i = e^{i\left(\frac{\pi}{2}+2n\pi\right)}$, for $n \in \mathbb{Z}$.

Therefore $z = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}n\right)}$, for $n = 0, 1, 2$.

For $n = 0$ we have $z_0 = e^{i\,\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2} + i\,\frac{1}{2}}$.

For $n = 1$ we have $z_1 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = e^{i\,\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \boxed{-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}}$.

For $n = 2$ we have $z_2 = e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = e^{i\,\frac{9\pi}{6}} = e^{i\,\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = \boxed{-i}$.

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Q-2) Calculate $\dfrac{(\sqrt{3}+i)^{67}}{(1+i)^{113}}$. Write your answer in the rectangular form $x+iy$, where x and y are real numbers.

Solution:

We first note that $(\sqrt{3}+i) = 2\, e^{i\,\frac{\pi}{6}}$, and $(1+i) = 2^{1/2}\, e^{i\,\frac{\pi}{4}}$. Hence we have

$$(\sqrt{3}+i)^{67} = \left(2\, e^{i\,\frac{\pi}{6}}\right)^{67} = 2^{67}\, e^{i\,\frac{67\pi}{6}} = 2^{67}\, e^{i(5\cdot 2\pi+\pi+\frac{\pi}{6})} = -2^{67}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = -2^{66}(\sqrt{3}+i).$$

Similarly we have

$$(1+i)^{113} = \left(2^{\frac{1}{2}}\, e^{i\,\frac{\pi}{4}}\right)^{113} = 2^{\frac{113}{2}}\, e^{i\,\frac{113\pi}{4}} = 2^{\frac{113}{2}}\, e^{i\,(14\cdot 2\pi+\frac{\pi}{4})} = 2^{\frac{113}{2}}\left(\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right) = 2^{56}(1+i).$$

Finally we have

$$\frac{(\sqrt{3}+i)^{67}}{(1+i)^{113}} = \frac{-2^{66}(\sqrt{3}+i)}{2^{56}(1+i)} = -2^{10}\,\frac{\sqrt{3}+1}{1+i} = -2^{10}\,\frac{\sqrt{3}+1}{1+i}\,\frac{1-i}{1-i} = -2^{9}\,(\sqrt{3}+1) + i\,2^{9}\,(\sqrt{3}-1).$$

This is approximately equal to $(-1398.81+374.81\,i)$. In polar coordinates we have $r \approx 1448.15$ and $\theta \approx 165°$.

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Q-3) Calculate the principal value of $(1+i)^{(1+i)}$. Write your answer in the rectangular form $x+iy$, where x and y are real numbers.

Solution:

The principal value corresponds in using sheet 0 for the logarithm function. In particular we have

$$(1+i)^{(1+i)} = \exp[(1+i)\log(1+i)] = \exp[(1+i)\log(\sqrt{2}\,e^{i\frac{\pi}{4}})] = \exp[(1+i)(\ln\sqrt{2} + i\,\frac{\pi}{4})].$$

Multiplying out we have

$$(1+i)^{(1+i)} = \exp[(\ln\sqrt{2}-\frac{\pi}{4})+i(\ln\sqrt{2}+\frac{\pi}{4})] = \exp[\ln\sqrt{2}-\frac{\pi}{4}]\cos(\ln\sqrt{2}+\frac{\pi}{4})+i\ \exp[\ln\sqrt{2}-\frac{\pi}{4}]\sin(\ln\sqrt{2}+\frac{\pi}{4}).$$

This is approximately $(0.27 + 0.58\,i)$. In polar coordinates we have $r \approx 0.64$ and $\theta \approx 65°$.

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Q-4) Consider the function $f(z) = \cosh x \cos y + i \sinh x \sin y$ from the $z = x + iy$ plane to the $w = u + iv$ plane. Describe the image of the following lines under the action of f.

(a) $y = 0$.

(b) $y = \pi/2$.

(c) $y = \pi$.

(d) $y = y_0$, where $0 < y_0 < \pi/2$.

(e) $y = y_0$, where $\pi/2 < y_0 < \pi$.

Solution:

We have $u = \cosh x \cos y$ and $v = \sinh x \sin y$. When $y = 0$, we see that f maps the x-axis to the u axis, but we must have $u \geq 1$. A careful analysis shows that we must have a cut in the w-plane along the u-axis from $u = 1$ to $+\infty$. The line $x \geq 0$ is mapped to the upper part of the cut, and the $x \leq 0$ is mapped to the lower part.

Similarly we cut the w-plane along the u-axis from $u = -1$ to $-\infty$, and the line $y = \pi,\ x \geq 0$ is mapped to the upper part of the cut while the line $y = \pi,\ x \leq 0$ is mapped to the lower part.

The line $y = \pi/2$ is mapped in a one-to-one onto manner to the v-axis.

When $y = y_0$ and $y_0 \neq 0, \pi/2, \pi$, we can write $\left(\frac{u}{\cos y_0}\right)^2 - \left(\frac{v}{\sin y_0}\right)^2 = 1$. This is the equation of a hyperbola which intersects the u-axis orthogonally at $(\cos y_0, 0)$. The hyperbola lies in the $u > 0$ half-plane if $0 < y_0 < \pi/2$, and on the $u < 0$ half-plane if $\pi/2 < y_0 < \pi$.

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Q-5) The cross-ratio of four distinct numbers $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$ is the image $T(z_4)$, where $T(z) = \dfrac{az+b}{cz+d}$ is the Mobius transformation mapping z_1 to 0, z_2 to 1 and z_3 to ∞. Here $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

(a) Find the cross-ratio of $0, 1, \infty, 57 + 89\,i$.

(b) Find the cross-ratio of $1, 2, 3, i$.

Solution:

(a) If $T(0) = 0,\ T(1) = 1$ and $T(\infty) = \infty$, then T is the identity transformation since it fixes three points. Then $T(57 + 89\,i) = 57 + 89\,i$.

(b) The transformation T which gives the cross-ratio of z, z_1, z_2, z_3 is given by

$$T(z) = \frac{z - z_1}{z - z_3}\,\frac{z_2 - z_3}{z_2 - z_1}.$$

In our case we have $z_1 = 1,\ z_2 = 2,\ z_3 = 3$. So we have

$$T(z) = \frac{z-1}{z-3}\,\frac{2-3}{2-1} = -\frac{z-1}{z-3}.$$

The required cross-ratio is then

$$T(i) = -\frac{i-1}{i-3} = -\frac{(i-1)}{(i-3)}\,\frac{(-i-3)}{(-i-3)} = -\frac{2}{5} + i\,\frac{1}{5}.$$