Due Date: 28 November 2015, Saturday
Time: 10:00-12:00
Instructor: Ali Sinan Sertöz

NAME:..
STUDENT NO:.......................................

Math 202 Complex Analysis – Midterm Exam II – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are **5** questions on your exam booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Cauchy-Riemann Equations: If $f(z) = u(x, y) + iv(x, y)$ is holomorphic on a domain, then at every point of that domain we have

$$u_x = v_y, \quad \text{and} \quad u_y = -v_x.$$

In polar coordinates, taking $z = x + iy = re^{i\theta}$, these equations have the form

$$ru_r = v_\theta, \quad \text{and} \quad u_\theta = -rv_r.$$

Moreover we have

$$f'(z) = u_x + iv_x = e^{-i\theta}(u_r + iv_r).$$

Cauchy-Goursat Theorem: If f is holomorphic on and inside of the closed contour C, then

$$\int_C f(z)\, dz = 0.$$

Cauchy Integral Formula: If f is holomorphic on and inside of the closed contour C, and if z_0 lies inside C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Moreover, if n is a positive integer, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Geometric Series: $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ for $|z| < 1$.

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Q-1) The transformation T which gives the cross-ratio of z, z_1, z_2, z_3 is given by

$$T(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}.$$

(a) Calculate $T(1 + i,\ -2i,\ 1,\ 4i)$.

(b) Do the points $1 + i,\ -2i,\ 1,\ 4i$ lie on a circle?

Solution:

(a)

If you take $T(z; z_1, z_2, z_3) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$, then a straightforward calculation gives

$$T(1 + i,\ -2i,\ 1,\ 4i) = \frac{46}{25} + \frac{3}{25}\, i.$$

However, if you take $T(z_1, z_2, z_3; z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$, then a straightforward calculation gives

$$T(1 + i,\ -2i,\ 1,\ 4i) = \frac{46}{85} - \frac{3}{85}\, i = \left(\frac{46}{25} + \frac{3}{25}\, i\right)^{-1}.$$

(b)

Four points lie on a circle if and only if their cross-ratio is real. Here the cross-ratio is not real, so these points do not lie on a circle. (No need to do geometric investigation once this cross-ratio is calculated!)

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Q-2) Calculate the following numbers and write your answer in the rectangular form, i.e. as $x + iy$.

(a) $\left(\frac{2}{\sqrt{2}} + i\,\frac{2}{\sqrt{2}}\right)^{2015}$

(b) $\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)^{2015}$

(c) $\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)^{2015}$

(d) i^{2015}

Solution:

(a)
Note that $2015 = 8 \times 251 + 7$. Hence

$$\left(\frac{2}{\sqrt{2}} + i\,\frac{2}{\sqrt{2}}\right)^{2015} = 2^{2015}\left(e^{i\,\frac{\pi}{4}}\right)^{8\times 251+7} = 2^{2015}e^{i\,\frac{7\pi}{4}} = \frac{2^{2015}}{\sqrt{2}} - i\,\frac{2^{2015}}{\sqrt{2}}.$$

(b)
Here $2015 = 12 \times 167 + 11$. Hence

$$\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)^{2015} = \left(e^{i\,\frac{\pi}{6}}\right)^{12\times 167+11} = e^{i\,\frac{11\pi}{6}} = \frac{\sqrt{3}}{2} - i\,\frac{1}{2}.$$

(c)
Here $2015 = 6 \times 335 + 5$. Hence

$$\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)^{2015} = \left(e^{i\,\frac{\pi}{3}}\right)^{6\times 335+5} = e^{i\,\frac{5\pi}{3}} = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

(d)
Here $2015 = 4 \times 503 + 3$. Hence

$$i^{2015} = i^{4\times 503+3} = i^3 = -i.$$

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Q-3) Evaluate the integral $\int_C \frac{z^2}{(z-1)(z-2)(z-3)}\, dz$, where C is the contour given below.

(a) $|z| = 1/2$.

(b) $|z| = 3/2$.

(c) $|z| = 5/2$.

(d) $|z| = 7/2$

Solution:

As a preparation for the solution set

$$f(z) = \frac{z^2}{(z-1)(z-2)(z-3)},\ f_1(z) = \frac{z^2}{(z-2)(z-3)},\ f_2(z) = \frac{z^2}{(z-1)(z-3)},\ f_3(z) = \frac{z^2}{(z-1)(z-2)}.$$

Moreover let C_k be the circle with center k and radius $1/2$, for $k = 1, 2, 3..$ Now set

$$J = \int_C \frac{z^2}{(z-1)(z-2)(z-3)}\, dz, \quad \text{and} \quad J_k = \int_{C_k} \frac{f_k(z)}{z-k}\, dz, \quad \text{for} \quad k = 1, 2, 3.$$

Note that we have

$$f(z) = \frac{f_k(z)}{z-k} \quad \text{for} \quad k = 1, 2, 3.$$

Now we are ready for the solution.

(a)
Inside $|z| = 1/2$, f is holomorphic, so by Cauchy-Goursat theorem we have $J = 0$.

(b)
Inside $|z| = 3/2$, the only singularity of f is at $z = 1$, so by Cauchy Integral Formula we have

$$J = J_1 = 2\pi i f_1(1) = (2\pi i)(\frac{1}{2}) = \pi i.$$

(c)
Inside $|z| = 5/2$, the singularities of f are at $z = 1$ and at $z = 2$, so by Cauchy Integral Formula we have

$$J = J_1 + J_2 = 2\pi i[f_1(1) + f_2(2)] = (2\pi i)[(\frac{1}{2}) + (-4)] = -7\pi i.$$

(d)
Inside $|z| = 7/2$, the singularities of f are at $z = 1$, $z = 2$and at $z = 3$, so by Cauchy Integral Formula we have

$$J = J_1 + J_2 + J_3 = 2\pi i[f_1(1) + f_2(2) + f_3(3)] = (2\pi i)[(\frac{1}{2}) + (-4) + \frac{9}{2}] = 2\pi i.$$

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Q-4) Write the Laurent series of $f(z) = \dfrac{1}{z^2 - 3z + 2}$ converging on the annulus $1 < |z| < 2$.

Solution:

First note that $1 < |z| < 2$ implies that $\left|\dfrac{z}{2}\right| < 1$ and $\left|\dfrac{1}{z}\right| < 1$. Next we observe that

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}.$$

Then we use geometric series to write

$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}.$$

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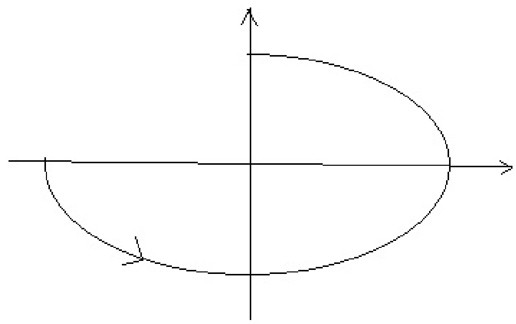
Q-5) Let C be the contour along the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ going from $z = -5$ to $z = 2i$, see figure below. Evaluate the following path integral.

$$\int_C z\,dz.$$



Solution:

Method 1

The integrand is analytic so the integral is path independent. The integral thus depends only at the end points. Using the Fundamental Theorem of Calculus for analytic functions we get

$$\int_C z\,dz = \left(\frac{z^2}{2} \bigg|_{-5}^{2i} \right) = -\frac{29}{2}.$$

Method 2

Since the integrand function z is holomorphic, its path integrals depend only at end points. So we choose a simpler path joining the points -5 and $2i$. Consider the path

$$z = t + i(\frac{2}{5}t + 2),\ t \in [-5, 0].$$

Then we have

$$z\,dz = (t + i(\frac{2}{5}t + 2)) \times (1 + \frac{2}{5}i)\,dt = \left[\left(\frac{21t - 20}{25} \right) + \left(\frac{4t + 10}{5} \right) i \right] dt.$$

Finally

$$\int_C z\,dz = \int_{-5}^{0} \left[\left(\frac{21t - 20}{25} \right) + \left(\frac{4t + 10}{5} \right) i \right] dt = -\frac{29}{2}.$$

Method 3

We directly use the definition of path integral to calculate the given integral.

$$z = 5\cos\theta + 2i\sin\theta \quad \text{for} \quad \theta \in [-\pi, \frac{\pi}{2}], \quad \text{and} \quad dz = (-5\sin\theta + 2i\cos\theta)\,d\theta.$$

Finally we get

$$\int_C z\,dz = \int_{-\pi}^{\pi/2} \left(-\frac{29}{2}\,\sin 2\theta + 10i\,\cos 2\theta \right) d\theta = \left([\frac{29}{4}\cos 2\theta + 5i\sin 2\theta] \bigg|_{-\pi}^{\pi/2} \right) = -\frac{29}{2}.$$