MATH 202 Complex Analysis
Homework 1
Solution Key

Show your work in reasonable detail. It is important that you explain your solution in a convincing way. The grader can but will not do mind reading!

1) Calculate the following and give your answer in rectangular form $a + ib$ where a and b are real numbers.

(a) All cubic roots of i.

(b) $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2021}$.

Solution:

Note that $\exp(i\pi/3) = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$.

Therefore $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2021} = \exp(i2021\pi/3) = \exp(i[673\pi + \frac{2\pi}{3}]) = \exp(i673\pi)\exp(\frac{i2\pi}{3}) = (-1)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

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2) Let $f(z) = u(x,y) + iv(x,y)$ be an entire function and let c_1, c_2 be two real numbers. Assume that the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ intersect at a point $z_0 = x_0 + iy_0$. Show that these two curves intersect at z_0 at right angles if $f'(z_0) \neq 0$.

Solution:

Let y_1 be the curve implicitly defined around z_0 by the equation $u(x, y_1) = c_1$. Similarly let y_2 be such that $v(x, y_2) = c_2$.

By chain rule we have at z_0

$$u_x + u_y\, y_1' = 0,$$
$$v_x + v_y\, y_2' = 0.$$

First assume that $u_y(z_0) \neq 0$ and $v_y(z_0) \neq 0$. Then at z_0 we will have

$$y_1' = -\frac{u_x}{u_y},\ y_2' = -\frac{v_x}{v_y},\ y_1'y_2' = \frac{u_x}{u_y}\frac{v_x}{v_y} = -1,$$

where the last equality follows from the Cauchy-Riemann equations.

Now suppose at z_0 we have $u_y = 0$. Then by Cauchy-Riemann equations we also have $v_x = 0$.

Since $f'(z_0) = u_x + iv_x = (1/i)(u_y + iv_y) \neq 0$, we must have $v_y \neq 0$ and $u_x \neq 0$. But $y_2' = -(v_x/v_y) = 0$ since $v_x = 0$. And from $u_x + u_y y_1' = 0$ we must have $y_1' = \pm\infty$ since $u_x \neq 0$. Hence the given level curves are again perpendicular, one being horizontal, the other vertical. Similarly if $v_y = 0$ at z_0.

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3) Let $f(z)$ be analytic at z_0, and $g(z)$ be analytic at $w_0 = f(z_0)$. Show that $(g \circ f)$(z) is analytic at z_0 and moreover show that $(g \circ f)'(z_0) = g'(w_0)\, f'(z_0)$.

Solution:

First assume that f is constant. Then $g \circ f$ is also constant hence trivially analytic everywhere and the required identity of the derivatives holds as both sides are now zero.

Now assume that f is not constant. Then the zero set of f has no accumulation point. In particular there is an open neighborhood U of z_0 such taht for all $z \in U$, $z \neq z_0$, we have $f(z) - f(z_0) \neq 0$.

Now we can try to see if $g \circ f$ has a derivative at z_0.

$$\begin{aligned}
(g \circ f)'(z_0) &= \lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} \\
&= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\
&= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
&= g'(f(z_0))\, f'(z_0).
\end{aligned}$$

Observe that for the limit of g' to make sense the denominator should never vanish unless $z = z_0$. This we guaranteed by taking f as a non-constant analytic function.

Now the above argument will work at every $z_1 \in U$ showing that $(g \circ f)'$ exists on U, hence $g \circ f$ is analytic at z_0.

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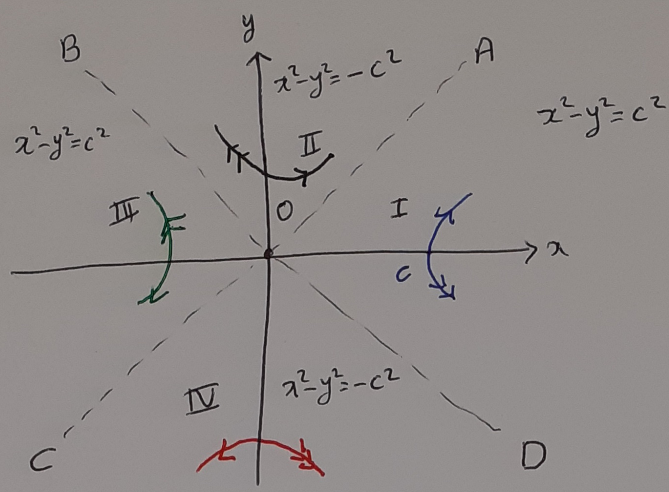
4) Consider the images of the hyperbolas $x^2 - y^2 = \pm c^2$ under the mapping $f(z) = z^2$, where $c > 0$. Show that you have two sheets on the image and show how these sheets are glued together so that f becomes one-to-one and onto this new surface.

Solution:

The map is $z \mapsto z^2 = (x^2 - y^2) + i(2xy)$. The following figures are self explanatory.



I: $x = c\cosh t$ $\quad u = c^2$
$y = c\sinh t$ $\quad v = c^2 \sinh 2t$

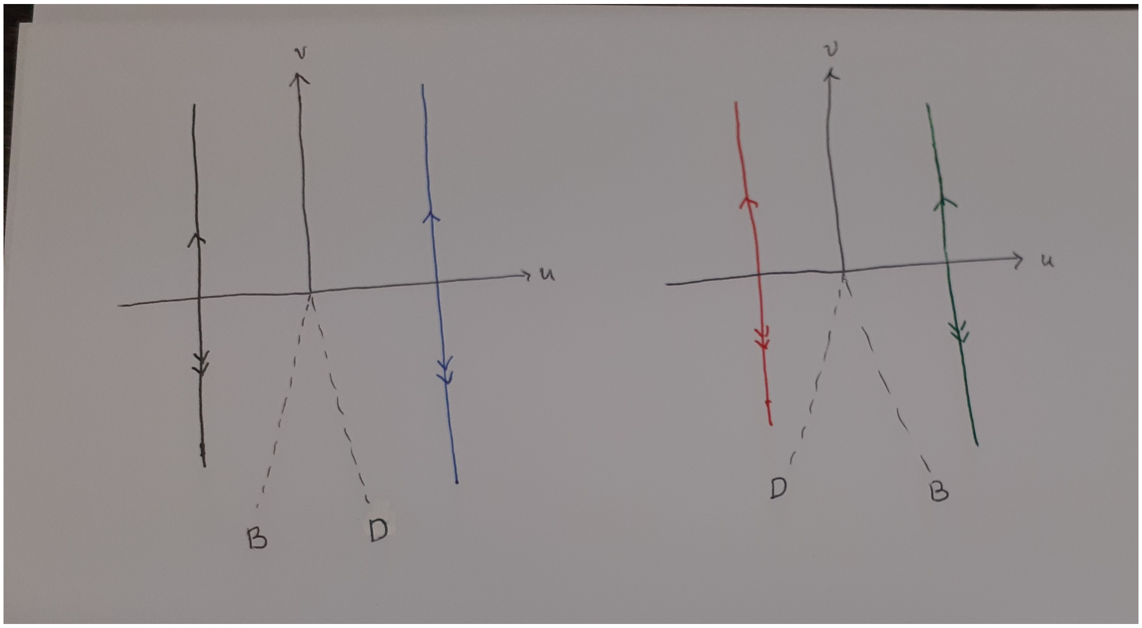
II: $x = c\sinh t$ $\quad u = -c^2$
$y = c\cosh t$ $\quad v = c^2 \sinh 2t$

III: $x = -c\cosh t$ $\quad u = c^2$
$y = -c\sinh t$ $\quad v = c^2 \sinh 2t$

IV: $x = -c\sinh t$ $\quad u = -c^2$
$y = -c\cosh t$ $\quad v = c^2 \sinh 2t$

$\rightarrow$ means we move in this direction as $t>0$ and $t\to\infty$

$\twoheadrightarrow$ means we move in this direction as $t<0$ and $t\to-\infty$.



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5) Calculate the principal value of $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^i$.

Solution:

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^i = \exp(-i\pi/3)^i = \exp(-i^2\pi/3) = \exp(\pi/3).$$

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